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INVESTIGATION OF CIVILIAN PROTECTED OBJECTS WITH THE USE OF FINITE ELEMENT MODELING METHOD

Abstract

In order to solve the tasks of civilian protection services, the method for modeling of the soil layer reaction under the engineering objects on the tectonic, seismic and other ecological and geophysical influences of the stress-strain state of the earth's crust in the engineering objects vicinity is developed. An algorithm for usage the finite element method is elaborated. The physical and mathematical formulation of the problem is carried out. Problem is solved on test examples. Solid models of engineering structures functioning for which the stress-deformation characteristics of the soil mass are simulated are developed. The theoretical determination and modeling of the critical values of the stress-strain state of the soil mass influenced by loads are carried out. The last allows predicting the nature of the influence of ecological-geophysical mechanical processes in the environment on the determination of stability of engineering structures. The stress-strain processes of the solid rock mass are explored, as a result of which there are extraordinary situations of natural and man-made nature.

Keywords: civil protection, finite element method, modeling, engineering objects

Introduction

In this paper the study of strain-stress state of the objects, which in general meaning must be protected with the participation of the civilian protection forses with the usage of finite elements method (FEM) approach is presented. The methods of services to avoid extraordinary situations in Poland and Ukraine are widely known [2, 3, 6, 8, 11]. Computer modeling of rock state by FEM is developed [1, 4, 5, 7, 9, 10]. Going out of this aim of the article is flow up is to combine two approaches and obtain FEM to be used by civilian protection forces.

The influence of various external factors such as natural disasters, among them influences that strain the medium near the bridges, roads, viaducts caused by massive rocks, which are in their turn in unstable condition because of meteorological and season events occurring in spatially delimitated situations while overloaded by train cars, lorries, unstable local parts of hill and mountain slopes create small rock vibrations. The last make local slippage of the parts of medium loaded that lead to the extraordinary situations on various parts of the inhabitable for people territories. Also in this row river moisture make influence on the banks of rivers that become unstable spatially in the periods of season changing with permanent influence on river bed curves (along or in the parts of their extension), in the vicinity of roads with heavy traffic. All this phenomena frequently lead to catastrophes and fires extremely dangerous in the summer hot periods of the year.

Nowadays, due to the development of the construction of engineering structures, as well as the reconstruction of already existed engineering structures, not only in Ukraine, but also in the whole world, the tendency is to a development of problems related to the rise of emergencies in engineering structures. According to statistics, only from 2013 and up to 2015, as a effect of the sudden destruction of buildings and structures in Ukraine, there were 11 accidents that led to a tragic conclusion, namely, the deaths of 14 people and 17 people were hospitalized, this is only human factor, and if we consider the ecology-geophysical number of accidents in the world it is abruptly rising. All of these emergency situations were the result of cracks, faults, the formation of so-called hollow holes (karst), raising the soil at the base of the joints of the soil array with a bridge structures, etc. The latter, in turn, is due to the excess of critical potential states of stress and deformation of the soil massif beneath lead to the destruction of the upper layers of overpasses and bridge structures.

The tasks of studying the crustal earth masses in the area of bridge-constructions using computer modeling methods are closely connected with the choice of the method for their solution. On the basis of the review of existing publications and methods of computer simulation, the choice has been made - to perform a study using the finite element method (FEM).

With FEM, it is possible to solve much more general problems in the study of processes occurring in the earth's crust than with the use of analytical methods: problems of diffraction on one or two inhomogeneities using the diffraction method and Born type approximation, solving by the matrix method, using Integral transformations, etc. [9]. The numeric method of finite differences gives an insight into the stress-strain state of the inhomogeneous earth's crust, but the use of this method involves inaccuracies in the solution associated with abrupt changes in the physical characteristics of the earth's crust: fractures, voids, displacements. The same in different manifestations may apply to combined methods.

The chosen FEM must be optimally used: to correctly carry out the physical and mathematical formulation of the problem, to develop an algorithm for using the method. If this is a numerical method (numerical methods, for the most part, provide an adequate solution to the problems posed by the development of modern science and computer technology), it is necessary to investigate the stability and convergence of the FEM, to solve tasks on test cases and to compare with practical results what is poaaible with use of elaborated method.

To resolve the listed tasks, the sections of this work are devoted.

From upper mentioned information follows the purpose of the article - evidence of necessity to obtain possibility to calculate (estimate more exactly) deformations and stresses especially in the vitally important parts of settlements infrastructure, notably in technologically intensively developed regions with objective to outline boundaries of weakened parts of rocks, study their geophysical nature (i.e. types of rocks nature or artificial: concrete frame, logs etc.), pressure and shear wave velocities or Young and Poisson ratios, densities, damping coefficients and finally recalculate the stress-strain state of the media utilizing interactive computer modeling for the each specific situation.

This work was fulfilled partly in the frame of the common project between Main School of Fire Service, Warsaw and L’viv State University of Life Safety and moderately given in seminar for teachers and

students during authors making lectures in Warsaw. Student of L'viv State University of Life Safety Vitaly Batyuk took part in computer simulations fulfilled in this article.

Method of finite element modeling for investigation of civilian protected objects

One of important problems in the design and construction of bridges, viaducts, tunnels for various purposes is forecasting ecological and geophysical character of the mechanical conditions of the adjacent massif in order to determine the stability and durability of bridge type structures (Fig.1).

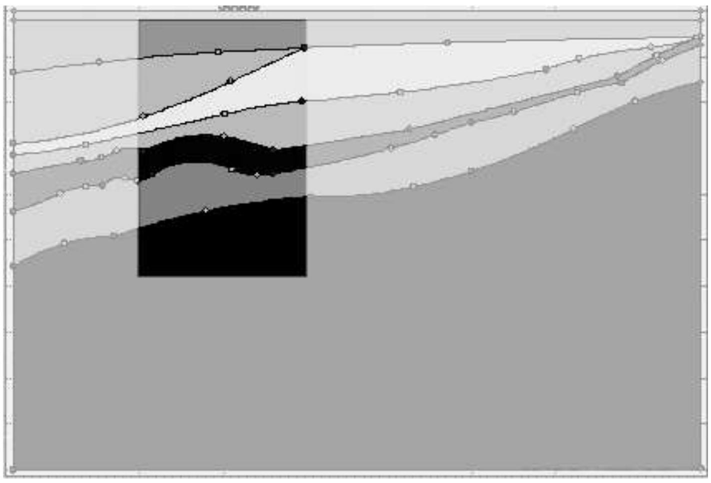

Fig.1. Model of the inhomogeneous half-space horizontal cross-section (shaded), where on the surface bridges type structure is placed
Source: Own study

Stress-strain state of soils and bridge structures is determined by geometrical properties with different deformation characteristics, distribution and on the basis of stress-strain state of bridges type structures, affect the strength of soil, depth of foundations layering of bridge structures, processes of soil freezing and melting, the supplies of the groundwater level and more. At present time the situation of bridge type structures state prediction based on numerical experiments utilizing the finite element method is used [1, 7, 9]. In this paper the methodology of mathematical modeling, designed to solve the problem of civil defense forces activity is elaborated. It should match the character of the stressed state of the soil surrounding the object array, the specific of construction, it displacement and vibration behavior.

In this regard we can treat the movement of the medium with the help of Newton's law. Summarizing displacements for all particles of the medium over certain volume (V) with external area (S) with density depending on a nature of substance ρ(r), where r is treated as a function of coordinates (Cartesian coordinates X,Y,Z as an example - r(x,y,z).

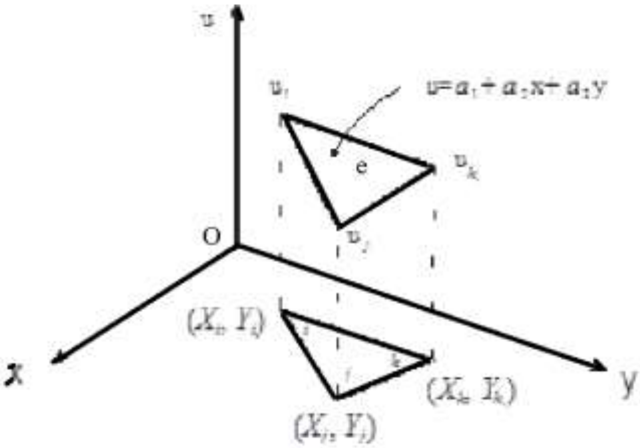

Fig.2. Model triangle element "e" cut of the inhomogeneous half-space horizontal cross-section XOY, where displacement **u** is shown over the XOY plane
Source: Own study

Thus taking into account that according to Newton's law for movement of particles with density $\rho(r)$ and acceleration d^2u/dt^2 in the medium depend on force we can write the following

$$\int_V \rho(r)\frac{d^2\mathbf{u}(r)}{dt^2}\mathrm{dv} = \int_S \boldsymbol{\sigma}(r)\mathrm{ds} \quad , \tag{1}$$

where $\mathbf{u} = (\mathrm{u}^i, \mathrm{u}^j, \mathrm{u}^k)^T$ displacement of linear approximation in the discretized half-space with components in FEM (Fig.2);

^{T} - mean matrix transposition; $\boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{xy}, \sigma_{xz})^T$ - stress column-vector; $\rho(r), \mathbf{u}(r), \boldsymbol{\sigma}(r)$ depend on Cartesian coordinates X,Y,Z. According to Gauss's theorem [ ] we have $\int_S \boldsymbol{\sigma}(r)\mathrm{ds} = \int_V \frac{d\boldsymbol{\sigma}(r)}{dr^2}\mathrm{dv}$, and equation for a motion of a rock for the integrable values from (1) is obtainable in the form of Newton's law for a motion

$$\int_V \rho(r)\frac{d^2\mathbf{u}(r)}{dt^2}\mathrm{dv} = \int_V \frac{d\boldsymbol{\sigma}(r)}{dr^2}\mathrm{dv} \quad , \qquad \rho(r)\frac{d^2\mathbf{u}(r)}{dt^2} = \frac{d\boldsymbol{\sigma}(r)}{dr^2} \ . \tag{2}$$

The last approximation is valid for the chosen separate discrete elements. To these elements the medium is diverged. It is assumed that in the limits of each element geophysical properties are not different and those elements while vibrate as an ensemble describe the behavior of all volume.

Linear approximation of discretized half-space displacement components in FEM is written in matrix form as the sum on N_e elements, where **u** – the displacement vector in the medium, i, j, k = 1, …, N_e; N_e – number of finite elements of the mesh partitioning the model. And the form for displacement on element "e" is

$$\mathrm{U} = (\mathrm{u}_{3i-2},\ \mathrm{u}_{3i-1},\ \ \mathrm{u}_{3i},\ \ldots\ \mathrm{u}_{3j-2},\ \mathrm{u}_{3j-1},\ \ \mathrm{u}_{3j},\ \ldots\ \mathrm{u}_{3k-2},\ \ \mathrm{u}_{3k-1},\ \ \mathrm{u}_{3k})^T \tag{3}$$

Upper column vector consists of displacements on i, j, k triangle apexes (Fig.2).

Using this for the triangle "e" (Fig.2) the displacement on element "e" of the medium is written in FEM in the form [ ]

$$\mathrm{U} = N_i\mathrm{u}_i + N_j\mathrm{u}_{j,} + N_k\mathrm{u}_k, \tag{4}$$

where

$$N_i = \frac{1}{2A}\left[a_i + b_i x + c_i y\right], \quad \begin{cases} a_i = X_i Y_k - X_k Y_j, \\ b_i = Y_i - Y_k, \\ c_i = X_k - X_j; \end{cases} \quad N_j = \frac{1}{2A}\left[a_j + b_j x + c_j y\right], \quad \begin{cases} a_j = X_k Y_i - X_k Y_i, \\ b_j = Y_k - Y_i, \\ c_j = X_i - X_k; \end{cases}$$
$$N_k = \frac{1}{2A}\left[a_k + b_k x + c_k y\right], \quad \begin{cases} a_k = X_i Y_j - X_j Y_i, \\ b_k = Y_i - Y_j, \\ c_k = X_j - X_i; \end{cases} \quad 2A = \begin{vmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{vmatrix}; \quad \text{i, j, k = 1, ..., } \mathrm{N_e}, \tag{5}$$

$\mathrm{N_e}$ - number of finite elements of the mesh partitioning a model.

The relationship between strain and displacements in propose approach is written in matrix form:

$$\boldsymbol{\varepsilon} = \mathbf{B}^{(e)}\mathrm{U}\ , \tag{6}$$

where $\mathbf{B}^{(e)}$ is the differential operator matrix, . $\boldsymbol{\varepsilon}$ – in considered case is written in the matrix-vector form:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{xy} \end{bmatrix}^T, \quad (7)$$

where $\varepsilon_{xx} = \frac{\partial u}{\partial x}$, $\varepsilon_{yy} = \frac{\partial v}{\partial y}$, $\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$.

Similarly equation for the stress (stress-strain state) of the medium (Hook's law) concerning considered cross-section has the form

$$\boldsymbol{\sigma} = \mathbf{D}^{(e)}\boldsymbol{\varepsilon}. \quad (8)$$

Here $\mathbf{D}^{(e)}$ - matrix of elastic properties of element (e), as previously determined $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{bmatrix}^T$.

Finite element method is based on the variation approach and consists in determining of the potential energy (Π) minimum for the studied medium

$$\delta\Pi = \delta\Lambda - \delta W, \quad (9)$$

where according formulas (6), (8) we have for the element "e" (Fig.2)

$$\Lambda^{(e)} = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dv = \frac{1}{2}\int_V \mathbf{U}^T \mathbf{B}^{(e)T} \mathbf{D}^{(e)} \mathbf{B}^{(e)} \mathbf{U}\, dv \quad (10)$$

- strain energy of the system, $\boldsymbol{\varepsilon}^{\mathbf{T}}, \boldsymbol{\sigma}$ - transposed columns of strain vector and column stress vector consequently;

$$W = W_B + W_P + W_C = \int_{V^{(e)}} \mathbf{U}^T \mathbf{N}^{(e)T} \mathbf{F}\, dv + \int_{S^{(e)}} \mathbf{U}^T \mathbf{N}^{(e)T} \mathbf{P}_2\, ds + \mathbf{U}^T \mathbf{P}_1, \quad (11)$$

where W - work of external forces consists of works of W_B body forces - $\mathbf{F}$, W_P surface forces - $\mathbf{P}_2$, W_C concentrated in nodes surface forces - $\mathbf{P}_1$;

$\mathbf{U}^T$ - transposed matrix-vector of linear approximation of discretized half-space displacement components in FEM written as the sum of influences of N_e elements

$$\mathbf{u} = (\mathrm{u}^i, \mathrm{u}^j, \mathrm{u}^k)^T, \quad \mathbf{u} = \mathbf{N}^{(e)}\mathbf{U}, \quad (12)$$

where $\mathbf{N}^{(e)} = \begin{bmatrix} N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 \\ 0 & N_i & 0 \ldots 0 & N_j & 0 \ldots 0 & N_k & 0 \\ 0 & 0 & N_i & 0 & 0 & N_j & 0 & 0 & N_k \end{bmatrix}$ - matrix that make connection between displacements in the element apexes and other points of modelled object (see formulae (5)).

Utilizing the expression for minimum of potential energy (9) is done in the following way. Summarizing expressions for the strain energy (10) and work of external forces (11) of the system and on N_e elements, differentiating the expression (9) on vector column ($\mathbf{U}$) to obtain energy minimum for the potential energy (Π) we obtain the matrix presentation of FEM equation for the medium state as a system of linear algebraic equations for $\mathbf{U}$

$$\mathbf{K}\,\mathbf{U} = \mathbf{f}, \quad (13)$$

where FEM stiffness matrix has the form $\mathbf{K} = \sum_{e=1}^{E} \int_{V^{(e)}} \mathbf{B}^{(e)T} \mathbf{D}^{(e)} \mathbf{B}^{(e)}\, dv$, f – vector-column of outside applied, surface and point forces is $\mathbf{f}^{(e)} = \int_{V^{(e)}} \mathbf{N}^{(\mathbf{e})T}\mathbf{F}\, dv + \int_{S^{(e)}} \mathbf{N}^{(\mathbf{e})T}\mathbf{P}_2\, ds + \mathbf{P}_1$.

Inertial and dissipative forces distributed by volume, so they can be considered as a part of the body forces. Taking into account the inertial forces in the elementary volume according equation (1), (2) $\rho\ddot{\mathbf{U}}$ ("$\dot{}$" – mean derivative on time, $\ddot{\mathbf{U}}$ - vector of second derivatives of displacements on time – acceleration, ρ - density of the body) and the dissipative force $c\dot{\mathbf{U}}$ (c - attenuation per unit volume obtained experimentally) for the discretized model for N_e elements we have

$$\mathbf{f_i} = \mathbf{M}\ddot{\mathbf{U}},\quad \mathbf{M} = \sum_{e=1}^{E} \int_{V^{(e)}} \rho_e \mathbf{N}^{(e)T}\mathbf{N}^{(e)}\, dv,\quad \mathbf{f_D} = \mathbf{C}\dot{\mathbf{U}},\quad \mathbf{C} = \sum_{e=1}^{E} \int_{V^{(e)}} c_e N^{(e)T} N^{(e)} dv. \tag{14}$$

Taking into account that inertial $\mathbf{f_i}$ and dissipative $\mathbf{f_D}$ forces are part of external forces directed against motion (vectors must have "minus") and substituting the values for them in (14) into FEM equation of the state of the medium (13) we obtain the equation of motion in matrix form

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{f} \tag{15}$$

where as previously $\mathbf{f}$ is vector-column of external forces. In the stationery loading in the right side of equation (15) and stationary boundary conditions for (15) **M**=**C**=**Z, Z** is zero matrix.

The question now is how to use appropriate value for the force $\mathbf{f}$. Mathematical model used here describes the stress-strain state in soil massif under the load of bridge type construction. For the determining of the force we use approximation Mohr-Coulomb criterion, which gives a surface in the three-dimensional case – the so-called Drucker-Prager model [5, 12, 13]. Model specimen preparation is based on the following mathematical relationship for the loading force:

$$\mathbf{f} = 3\alpha\sigma_m + \sigma_{eqv} - k, \tag{16}$$

where $\sigma_m = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$ - hydrostatic pressure, $\sigma_{eqv} = \sqrt{0.5}\sqrt{(s_x^2 + s_y^2 + s_z^2)/2 + (\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)}$ - deviatoric stress with parameters: $s_x = \sigma_{xx} - \sigma_m,\ s_y = \sigma_{yy} - \sigma_m,\ s_z = \sigma_{zz} - \sigma_m$, ; $\alpha = \dfrac{2\sqrt{3}\cdot\sin\varphi}{3(3-\sin\varphi)},\ k = \dfrac{2\sqrt{3}\cdot c\cdot\cos\varphi}{(3-\sin\varphi)}$ - coefficients for calculating force in the expression (16). In the last formula c [Pa] is known as coefficient of cohesion, φ - angles is to be given in degrees, determines a friction.

In the elaborated method $\mathbf{f}$ is taken in the frame of the Drucker-Prager model. This presentation allows three-dimensional body to be simulated by two- dimensional model.

In the next chapter the modeling results are presented.

Results of computer simulations

Utilizing upper described algorithm step be step we will present the computer simulation experiment from data prepare to computer obtained results for the modelling of bridge type construction.

Data preparation: material properties of the soil used in the modelling experiment are as follows: Young's modulus - E = 10^6 Pa, Poisson ratio ν = 0.3, c = 10^3 Pa, φ = 35 deg. The soil weight per unit volume (specific weight) is γ = 18 10^3 N/m^3, coefficient of cohesion ***c***= 10^4 Pa. The bridge is considered perfectly elastic, the values of the material properties initially are: E = 25 10^9 Pa, ν = 0.33, γ = 25 10^3 N/m^3.

Limitations and loading. The lower horizontal boundary of the model is restricted from moving in a horizontal X, and vertical directions, thus simulating a rigid main bearing rock. The model includes only one half of the bridge due to symmetry axis on the right. Restrictions apply to the plane of symmetry within a symmetric cut. The left vertical boundary is perfectly smooth and rigid modeling constraints only to the horizontal direction, allowing movement in the vertical direction (Fig.3). Densities of soil are introduced as the load per unit volume (correspond to the negative direction of the Y axis).

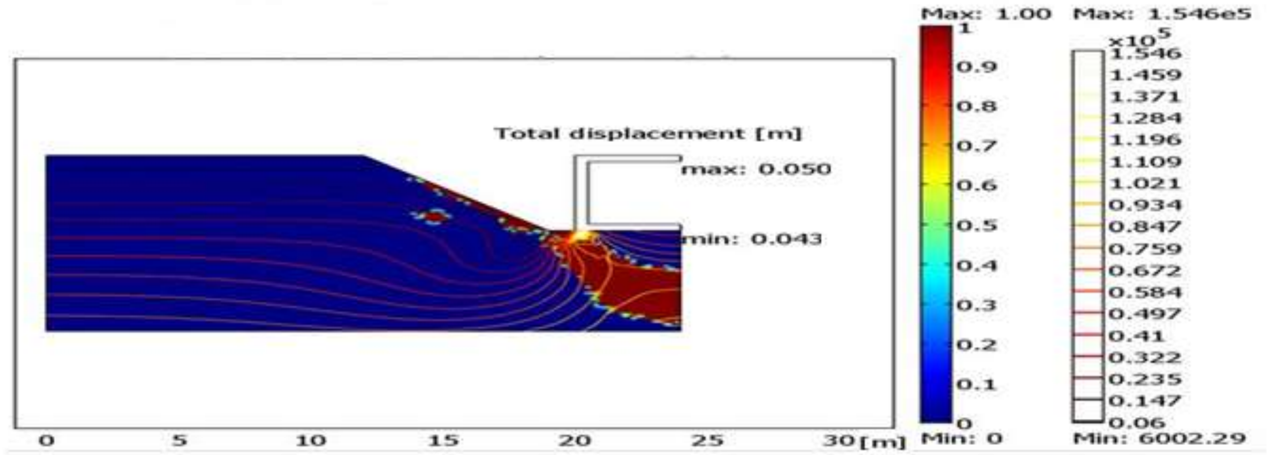

Fig.3 Cross-section of the modelled tunnel colors show of deformations in YX plane on the left (horizontal dimension from 0 to 30 m correspond to X axis, vertical dimension coincide with Y axis, scale the same as on horizontal axis). Total displacements are shown. Left column on the right depict deformation scale, right column – colors of stress isolines

Source: Own study

Weight per unit volume and strain are modelled as in natural state during elastic-plastic analysis method developed in the paper. Initially soil has some natural tension in the natural state (before loading):

$$\sigma_{xx} = \lambda\sigma_{yy}, \quad \sigma_{yy} = -\gamma y, \; \lambda = \frac{\nu}{(1-\nu)} .$$

With the usage of upper described material properties, limitations and loadings the results obtained for the models shown on the Fig4 are obtained. On the Figure 4A) the image of the bridge for which modelling carried out is shown. In addition to the bridge construction we pay attention to such factors as power lines, adjacent buildings and, most importantly, the presence of vehicles that are in the risk zone in the possible event of an accident. Figure 4A) depicts a bridge in the area of the Warsaw Street of Lviv, Ukraine.

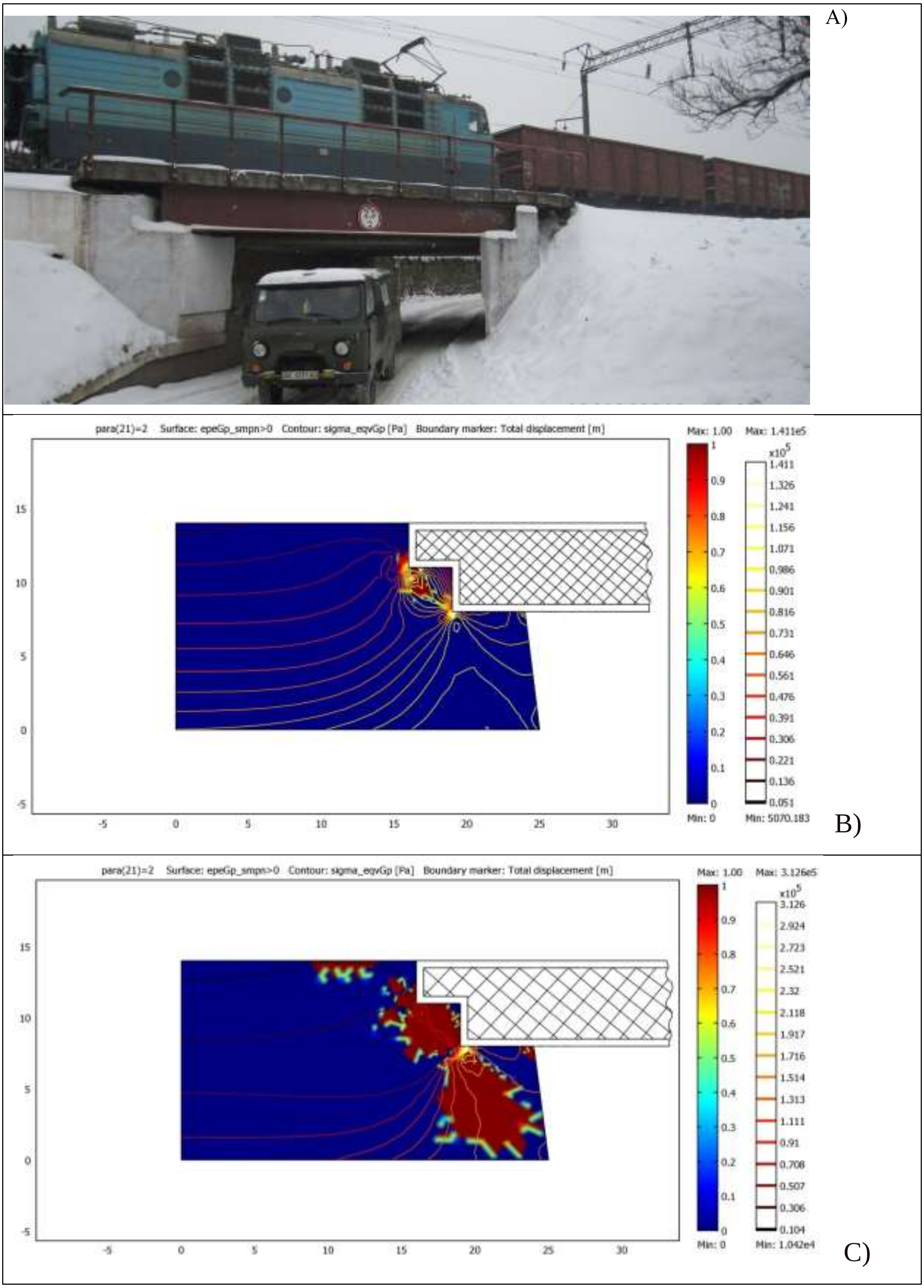

Fig.4. Modeling of rocks stress-strain state in the foundations of frame type structures. The example of, Warsaw street, Lviv; A) - the view of the sight; B) – strain – colors, isolines - stresses distribution for the

bridge construction model without loading, legends for strain and stress on the right, geophysical characteristics in the text; C) – the same model as B) overloaded by train
Source: Own study

The shape of the edge of the modeling bridge-engineering design is shown. Unloaded and loaded (by train weight) the shape of the bridge is visible in Figures 4B) and 4C), respectively. Also, in the background of Figure 4A), it is possible to see visible buildings - houses of people live located in the risk area in the event of an accident.

The behavior of natural model is simulated according to the bridge construction on Figure 4A), with the determined stress-strain state of the soil mass, the specific weight of which is 18 kN/m^3, the coefficient of cohesion 10 kPa, which corresponds to the physical and mechanical characteristics of loamy soil. In turn, the bridge-engineering structure is of specific weight 56 kN/m3, in accordance with the characteristics of reinforced concrete. The stress-strain state of the array under the influence of the loading of only the bridge-engineering structure is shown, deformations and stresses are observed, which are concentrated uniformly, only on the lateral area adjacent to the body of the bridge and the soil massif. Considering the change in the stresses and deformations of an identical model of the same bridge-engineering structure, with only an additional load for example, the theoretical calculated weight of the train is taken (for a single freight wagon of about 75 tons, we took half as the load is distributed along the whole length, that is, 32.5 tons) moving along a bridge-engineering structure is seen on Figure 4C). We observe a clear change in the stresses (increasing stress more than two times) and a significant increase in the deformation, and the main thing to note is that the deformation extends to the surface of the bridge construction, which indicates the possibility of negative factors formation in the sphere of loading influence, and the need for its strengthening.

The **practical and scientific novelty** of the FEM usage investigation for civilian protection

The elaborated method and obtained results of the stress-strain state study of the objects (bridge type construction under civilian protection) allow control of geotechnical processes in them, specifically strengthening supports of them in order to reduce the negative impact of man-made environmental factors and catastrophes. Also, based on our research, it is theoretically possible for civilian guards to calculate and select the critically possible properties of a bridge structures for an existing type of the rock with certain geophysical characteristics of their stress-strain behavior.

Conclusions

In the result of carrying out a series of experiments, model results obtained that enable calculate and determine the critical permissible stress-strain states of the rock massif of constructions for civilian defense units, and thus prevent the destructions as a result of excessive stresses and deformations arise under the influence of loads. Based on the research, it is theoretically possible to calculate and select the critically possible properties of a bridge construction for an existing type of soil mass with certain geophysical characteristics.

Application of the above methodology allows managing the geotechnical processes in order to reduce the negative impact of man-made factors on the environment. It gives the possibility to theoretically predict the stability and durability of engineering structures and the whole systems as a whole.

References

[1] Bathe K.-J. The Finite Element Procedures. – Prentice Hall, 2006. – 1037 p.

[2] Cecilia J., Alistair D.B. Cook Civilian Protection in the Twenty-First Century. – Oxford University Press, 2016. – 280 p.

[3] Chun'An Tang, John A. Hudson Rock Failure Mechanisms: Illustrated and Explained. - CRC Press 2011. - 319 p.

[4] Hjelmstad, Keith D. Fundamentals of Structural Mechanics 2nd ed., 2005, XIV, 480 p.

[5] Starodub George Modeling of rocks stress-strain state in the foundations of frame type structures // Geomatyka i inżynieria: kwartalnik naukowy PWSZ. – Jarosław, 2013. – № 2. – S. 21-25.

[6] Wang Sijing, Fu Bingjun, Li Zhonkui Frontiers of Rock Mechanics and Sustainable Development in the 21st Century. - CRC Press, 2001. - 650 p.

[7] Zienkiewicz O.C., TaylorR.L. The Finite Element Method for Solid and Structural Mechanics. – Elsivier, 2005. - 648 p.

[8] Іванік О. Головні особливості формування та активізації структурних осувів в басейні річки Латориця (Східні Карпати) Вісник Київського національного університету імені Тараса Шевченка 2009 13-18

[9] Стародуб Ю. П. Математичне моделювання в динамічних задачах сейсміки стосовно до вивчення будови земної кори: дисертація доктора фізико-математичних наук: 04.00.22 / Стародуб Юрій Петрович. — Інститут геофізики ім.С.І.Субботіна НАН України, Київ, 2002. – 273 с.

[10] Стародуб Ю.П. Методика моделювання хвильових полів і напружено-деформованого стану в неоднорідному півпросторі з використанням методу скінченних елементів / Стародуб Ю.П., Кендзера О.В., Купльовський Б.Є.// Препр. / НАН України, Центр математичного моделювання ІППММ; №7, 2007. – 44 с.

[11] Трофимчук А.Н. Математическое моделирование устойчивости оползневого склона при подъеме уровня грунтовых вод / А.Н. Трофимчук, Ю.И. Калюх, А.С. Глебчук // Екологія і ресурси. – 2008. – № 18. – С. 51-58.

[12] Wikipedia - Drucker–Prager yield criterion

[13] Wikipedia - Mohr–Coulomb theory