General Scheme of Modeling of Longitudinal Oscillations in Horizontal Rods

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Abstract. In this paper, we present the results of modeling nonstationary oscillatory processes in rods consisting of an arbitrary number of pieces. When modeling oscillatory processes that occur in many technical objects (automotive shafts, rods) an important role is played by finding the amplitude and frequency of oscillations. Solving oscillatory problems is associated with various difficulties. Such difficulties are a consequence of the application of methods of operation calculus and methods of approximate calculations. The method of modeling of oscillatory processes offered in work is executed without application of operational methods and methods of approximate calculations. The method of oscillation process modeling proposed in this paper is a universal method. The work is based on the concept of quasi-derivatives. Applying the concept of quasi-derivatives helps to avoid the problem of multiplication of generalized functions. Analytical formulas for describing oscillatory processes in rods consisting of an arbitrary number of pieces are obtained. It can be applied in cases where pieces of rods consist of different materials, and also when in places of joints the masses are concentrated. The proposed method allows the use of computational software. An example of constructing eigenvalues and eigenfunctions for a rod consisting of two pieces is given.

Keywords: Kvazidifferential equation · The boundary value problem · The cauchy matrix · The eigenvalues problem · The method of fourier and the method of eigenfunctions

1 Introduction

The problem of finding eigenvalues and eigenfunctions for equations in partial derivatives of the second order is an urgent problem. The relevance is due to the fact that such problems arise in the modeling of oscillatory processes of many technical systems. Each specific model is a separate mathematical problem, the ability to solve which depends on the input conditions. The theory of oscillatory processes is described in detail in [2,4,15,17]. In these works the mathematical and physical bases of oscillatory processes and methods of their modeling are stated. The method of solving boundary value problems according to which the solution of

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problems is reduced to solving simpler problems is called the method of reduction. The solution of the general boundary value problem is sought in the form of the sum of functions, one of which is the solution of a homogeneous problem, and the other is an inhomogeneous problem with zero boundary conditions. One of the most common methods for solving inhomogeneous problems is the method of separating variables, or the Fourier method. According to this method, the solution of a homogeneous problem is sought in the form of the product of two functions of one variable. Such a problem is called a problem of eigenvalues and eigenfunctions. It is also called the task of Shturm Liouville [1,9,11]. The Fourier method is an accurate method of solving these problems. In the process of solving problems by this method there are problems with the justification of the convergence of series and the multiplication of generalized functions [6,10,19]. In some cases, these problems can be avoided by reducing the system to a matrix form by introducing the so-called quasi-derivative [4,7,13]. In this paper, we used this method to find solutions to four problems of oscillatory processes and demonstrated the possibility of finding the required number of eigenvalues and eigenfunctions. This method is new. It allows you to avoid multiplication of generalized functions. The proposed method has an advantage over other methods in that it allows the use of computational mathematical packages.

2 Problem Statement

Oscillation processes are modeled using hyperbolic type equations. Quite often, it is almost impossible to obtain closed-loop solutions of such differential equations. The method proposed in our work belongs to the direct methods of solving boundary value problems, as a result of which the solutions are obtained in a closed form. A feature of our work is the use of a quasi-derivative. This approach will make it possible to write equations with partial derivatives of the second order of the hyperbolic type with general boundary conditions to the matrix form. The process of constructing the solution is based on the multiplication of matrices. Consider a second-order differential equation in partial derivatives

$$m(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(\lambda(x)\frac{\partial u}{\partial x}) \tag{1}$$

where $x \in (x_0; x_n)$, $t \in (0; +\infty)$

Denote the product $\lambda(x)\frac{\partial u}{\partial x}$ by $u^{[1]}$ and let's call it quasi-derivative. Write down the general boundary conditions:

$$\begin{cases} p_{11}u(x_0;t) + p_{12}u^{[1]}(x_0;t) + q_{11}u(x_n;t) + q_{12}u^{[1]}(x_n;t) = \psi_0(t) \\ p_{21}u(x_0;t) + p_{22}u^{[1]}(x_0;t) + q_{21}u(x_n;t) + q_{22}u^{[1]}(x_n;t) = \psi_n(t) \end{cases} \tag{2}$$

The initial conditions are the following:

$$\begin{cases} u(x;0) = \Phi_0(x) \\ \frac{\partial u}{\partial t}(x;0) = \Phi_1(x) \end{cases} \tag{3}$$

where $\psi_0(t), \psi_n(t) \in C^2(0;+\infty)$, $\Phi_0(x), \Phi_1(x)$ are piecewise continuous on $[x_0; x_n]$

3 Literature Review

As defined in the introduction, the problem of finding eigenvalues and eigenfunctions of a level in private derivatives of other hyperbolic conditions is an urgent problem in modeling quantitative processes of different technical systems. In the article [8] the case of rod oscillations under the action of periodic force is considered. What is special is that the action of force is distributed along the rod at a certain speed. Finding eigenvalues and eigenfunctions is a key task for solving the equation with second-order partial derivatives. The work [7] is devoted to finding a class of self-adjoint regular eigenfunctions and eigenvalues for each natural n. The boundary value problem problem for the hyperbolic equation in the rectangular domain is considered in [12]. A feature of this work is the case of singular coefficients of the equation. A very important aspect is the proof of the existence and uniqueness of the solution (Cauchy problem) and the proof of the stability theorem of the solution. The solution is obtained in the form of a Fourier-Bessel series. In the article [16] the matrix approach to the decision of a problem on eigenvalues and eigenfunctions (Sturm-Liouville problem) of the equation of hyperbolic type is offered. Emphasis is placed on the fact that the type of solution depends on the structure of the matrices. The matrix approach to the problem presentation is very convenient for the use of computational software. In our case the method of matrix calculus and the presentation of the differential equation in partial derivatives and the most general boundary conditions in matrix form are also used. In [3] a two-step method of discretization of a combined hyperbolic-parabolic problem with a nonlocal boundary condition was proposed. Examples of solving such problems by numerical methods are given. The problem of stability of solutions of the second-order problem considered in the work [18]. The method of introducing a quasi-derivative and reducing the equation of thermal conductivity to the matrix form is used in the work [13]. However, despite the achievements in this subject area, the problem of series convergence, multiplication of generalized functions and obtaining solutions of equations in partial derivatives remains open. The solution to this problem can be achieved to some extent through the use of modern calculation technologies using software tools [5]. The authors proposed to use the matrix form of the second-order differential equation and modern methods of calculations using mathematical software to model the oscillatory processes in the works [14]. The objective of the research is to find solutions of the equation of rod oscillations with different load and conditions using the concept of a quasi-derivative.

4 The General Scheme of Search of the Solution

The general scheme of finding a solution is to build two functions $w(x,t)$ and $v(x,t)$ such as

$$u(x,t) = w(x,t) + v(x,t)$$

Let's find the function $w(x,t)$ by a constructive method, then the function $v(x,t)$ will be defined unambiguously. The function $w(x,t)$ is a solution of a boundary value problem

$$(\lambda(x){w_x}')'_x = -f(x) \tag{4}$$

with the boundary conditions (2). We reduce this problem to a matrix form

$$\overline{W'_x} = A(x)\cdot\overline{W} + \overline{F} \tag{5}$$

$$P\cdot\overline{W}(x_0,t) + Q\cdot\overline{W}(x_n,t) = \overline{\Gamma}(t) \tag{6}$$

Where

$$A(x) = \begin{pmatrix} 0 & \frac{1}{\lambda(x)} \\ 0 & 0 \end{pmatrix}, \overline{W} = \begin{pmatrix} w \\ w^{[1]} \end{pmatrix}, \overline{F} = \begin{pmatrix} 0 \\ -f(x) \end{pmatrix},$$

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \overline{\Gamma}(t) = \begin{pmatrix} \psi_0(t) \\ \psi_1(t) \end{pmatrix}$$

Let $x_0 < x_1 < x_2 <...< x_{j-1} < x_j < x_{j+1} <...< x_{n-1} < x_n$ are the arbitrary partition of the segment $[x_0; x_n]$ of the real axis Ox into n parts. The solution of the problem at each interval $[x_i; x_{i+1})$ is the following:

$$\overline{W}_i(x,t) = B_i(x,x_i)\cdot\overline{P}_i + \int_{x_i}^{x} B_i(x,x_s)\cdot\overline{P}_i(s)ds$$

where $B_i(x,s) = \begin{pmatrix} 1 & b_i(x,s) \\ 0 & 1 \end{pmatrix}$ is the Cauchy matrix of a system, $b_i(x,s) = \int_s^x \frac{1}{\lambda_i(z)}dz$ Let's build matrices (for an arbitrary $k \geqslant i$)

$$B(x_k,x_i) = \begin{pmatrix} 1 & \sum_{m=i}^{k-1} b_m(x_{m+1},x_m) \\ 0 & 1 \end{pmatrix}$$

We use the recurrent method of mathematical induction to construct vectors $\overline{P_i}$

$$\overline{P_i} = B(x_i,x_0)\cdot\overline{P_0} + \sum_{k=0}^{i} B(x_i,x_k)\overline{Z_k},$$

where $\overline{Z}_k = \int_{x_{k-1}}^{x_k} B_{k-1}(x_k,s)\cdot\overline{F}_{k-1}(s)ds$. The vector $\overline{P}_0$ determines from the initial conditions by the formula:

$$\overline{P}_0 = [P + Q\cdot B(x_n,x_0)]^{-1}\cdot(\overline{\Gamma} - Q\sum_{k=1}^{n} B(x_n,x_k)\overline{Z}_k)$$

The first coordinate of the vector $\overline{W_i}(x,t)$ is indeed the searched function $w_i(x,t)$. Thus function $w(x,t)$ is a sum $w(x,t) = \sum_{i=0}^{n-1} w_i(x,t)\theta_i$, θ_i is the characteristic function of the interval $[x_i; x_{i+1})$, that is $\theta_i(x) = \begin{cases} 1, x\in[x_i,x_{i+1}), \\ 0, x\notin[x_i,x_{i+1}), \end{cases} i = \overline{0,n-1}$.
The function $v(x,t)$ is the solution of a mixed problem

$$m(x)\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}(\lambda(x)\frac{\partial v}{\partial x}) = -m(x)\frac{\partial^2 w}{\partial t^2} \tag{7}$$

Initial conditions for the function $v(x,t)$ are the following:

$$\begin{cases} v(x,0) = \Phi_0(x), \\ \frac{\partial v}{\partial t}(x,0) = \Phi_1(x), \end{cases} \tag{8}$$

where

$$\Phi_0 \stackrel{def}{=} \phi_0(x) - w(x,0)$$

$$\Phi_1 \stackrel{def}{=} \phi_1(x) - \frac{\partial w}{\partial t}(x,0).$$

The boundary conditions for the function $v(x,t)$ will be the following:

$$\begin{cases} p_{11}v(x_0) + p_{12}v^{[1]}(x_0) = 0 \\ q_{21}v(x_n) + q_{22}v^{[1]}(x_n) = 0 \end{cases} \tag{9}$$

We build the function $v(x,t)$ in the form of a product

$$v(x,t) = T(t) \cdot X(x)$$

Solve the problem of eigenvalues and eigenfunctions

$$(\lambda(x)X'(x))' + \omega^2 m(x)X(x) = 0 \tag{10}$$

$$\begin{cases} X(x_0) = 0 \\ X(x_n) = 0 \end{cases} \tag{11}$$

Let's transform the problem into a matrix form. For this purposes let's denote

$$X[1] \stackrel{def}{=} \lambda X', \overline{X} = \begin{pmatrix} X \\ X^{[1]} \end{pmatrix}, A = \begin{pmatrix} 0 & \frac{1}{\lambda(x)} \\ -m(x)\omega^2 & 0 \end{pmatrix}.$$

Cauchy matrix

$$\widetilde{B}(x_n, x_0, \omega) \stackrel{def}{=} \begin{pmatrix} b_{11}(\omega) & b_{12}(\omega) \\ b_{21}(\omega) & b_{22}(\omega) \end{pmatrix}.$$

With this designations, the problem (10), (11) takes the form:

$$\overline{X}' = A \cdot X \tag{12}$$

$$P \cdot \overline{X}(x_0) + Q \cdot \overline{X}(x_n) = 0. \tag{13}$$

We solve the problem in each segment $[x_i; x_{i+1})$. The eigenfunctions $X_k(x, \omega_k)$ as the first coordinates of the eigenvectors $\overline{X}_k(x, \omega_k)$ can be written down as

$$X_k(x, \omega_k) = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \widetilde{B}(x, x_0, \omega_k) \cdot \overline{C} \tag{14}$$

In order to exist vector $\overline{C}$, there must be such condition

$$det(P + Q \cdot \widetilde{B}(x_n, x_0, \omega)) = 0 \tag{15}$$

The Eq. (14) is a characteristic equation of the problem (10), (11) or (12), (13). We will look for the function $v(x,t)$ by arranging it in a series by orthogonal functions $X_k(x,\omega_k)$, which are the solutions of the set of equations

$$v(x,t)=\sum_{k=1}^{n} T_k(t)\cdot X_k(x,\omega_k), \tag{16}$$

where $T_k(t)$ are the solutions of a set of equations

$$T_k''(t)+\omega_k^2\cdot T_k(t)=-w_k(t).$$

Thus, finally a solution of the mixed problem (7)–(8) is received in a form of the series

$$v(x,t)=\sum_{k=1}^{\infty}\left[\Phi_{0k}\cos\omega_k t+\frac{\Phi_{1k}}{\omega_k}\sin\omega_k t-\frac{1}{\omega_k}\int_0^t \sin\omega_k(t-s)\cdot w_k(s)ds\right]\cdot X_k(x,\omega_k). \tag{17}$$

For some function types consider how the method description works.

5 A Case of Piecewise Continuous Functions with Concentrated Masses

Let in (1) define $m(x)$ and $\lambda(x)$ - are piecewise continuous functions

$$m(x)=\sum_{i=0}^{n-1} m_i(x)\theta_i,\ m_i(x)>0,$$

$$\lambda(x)=\sum_{i=0}^{n-1}\lambda_i(x)\theta_i,\ \lambda_i(x)\in C[x_i,x_{i+1}),\ \lambda_i(x)>0,$$

$$f(x)=\sum_{i=0}^{n-1} g_i(x)\theta_i+\sum_{j=0}^{n-1} s_j\delta_j(x-x_j),\ g_i\in C[x_i,x_{i+1}),\ s_j\in\mathbb{R},$$

$$P=\begin{pmatrix}1\ 0\\0\ 0\end{pmatrix},Q=\begin{pmatrix}0\ 0\\1\ 0\end{pmatrix}.$$

By using the described method, we can avoid multiplication of generalized functions that are embedded in the function $f(x)$. Let's denote

$$\sigma_n=\sum_{m=0}^{n-1} b_m(x_{m+1},x_m),\ I_{k-1}(x_k)=-\int_{x_{k-1}}^{x_k} b_{k-1}(x_k,s)g_{k-1}(s)ds,$$

$$I_{k-1}^{[1]}(x_k)=-\int_{x_{k-1}}^{x_k} g_{k-1}(s)ds.$$

With such designations

$$w_i(x,t) = \psi_0(t) + (b_i(x,x_i) + \sigma_i) \cdot \frac{\psi_n(t) - \psi_0(t)}{\sigma_n}$$

$$-\frac{1}{\sigma_n}(b_i(x,x_i) + \sigma_i) \cdot \sum_{k=0}^{n}(I_{k-1}(x_k) + (I_{k-1}^{[1]}(x_k) - s_k) \cdot \sum_{m=k}^{n-1} b_m(x_{m+1}, x_m)$$

$$+\sum_{k=0}^{i}(I_{k-1}(x_k) + (I_{k-1}^{[1]}(x_k) - s_k) \sum_{m=k}^{i-1} b_m(x_{m+1}, x_m))$$

$$+b_i(x,x_i) \sum_{k=0}^{i}(I_{k-1}^{[1])}(x_k) - s_k)) + I_i(x)$$

Vector $\overline{C}$ in (14) is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The characteristic equation of function $v(x,t)$ takes the form $b_{12}(\omega) = 0$ and function $v(x,t)$ takes the form (17).

6 A Case of Piecewise Constant Functions with Concentrated Masses

Let in (1) $m(x)$, $f(x)$ – are piecewise constant with concentrated masses, $\lambda(x)$ – is piecewise constant:

$$m(x) = \sum_{i=0}^{n-1} m_i \theta_i(x) + \sum_{i=0}^{n-1} M_i \delta(x - x_i),$$

$$f(x) = \sum_{i=0}^{n-1} g_i \theta_i(x) + \sum_{i=0}^{n-1} S_i \delta(x - x_i),$$

and $\lambda(x)$ – are piecewise constant functions $\lambda(x) = \sum\limits_{i=0}^{n-1} \lambda_i \theta_i(x)$. In boundary conditions (2)

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In this case it is possible to specify the function $w_i(x,t)$

$$w_i(x,t) = \psi_0(t) + \left(\frac{x - x_i}{\lambda_i} + \sum_{m=0}^{i-1} \frac{\triangle x_{m+1}}{\lambda_m} \right) \cdot \frac{\psi_n(t) - \psi_0(t)}{\sum\limits_{m=0}^{n-1} \frac{\triangle x_{m+1}}{\lambda_m}}$$

$$-\frac{1}{\sum\limits_{m=0}^{n-1} \frac{\triangle x_{m+1}}{\lambda_m}} \cdot \left(\frac{x - x_i}{\lambda_i} + \sum_{m=0}^{i-1} \frac{\triangle x_{m+1}}{\lambda_m} \right) \cdot$$

$$\cdot\left(\sum_{k=1}^{n}\frac{g_{k-1}\triangle x_k^2}{2\lambda_{k-1}}-(g_{k-1}\triangle x_k+s_k)\sum_{m=k}^{n-1}\frac{\triangle x_{m-1}}{\lambda_m}\right)$$
$$+\sum_{k=1}^{i}\left(\frac{g_{k-1}\triangle x_k^2}{2\lambda_{k-1}}-(g_{k-1}\triangle x_k+s_k)\sum_{m=k}^{i-1}\frac{\triangle x_{m-1}}{\lambda_m}\right)$$
$$-\frac{x-x_i}{\lambda_i}\sum_{k=1}^{i}(g_{k-1}\triangle x_k+s_k)+\frac{g_i(x-x_i)^2}{2\lambda_i}$$

To build a function $v(x,t)$ we use the method of expansion by the eigenfunctions. Due to the delta functions in the left and right part of the equation, we go to the system

$$\overline{X}'=\left(\sum_{k=0}^{n-1}A_k\theta_k+\sum_{k=0}^{n-1}C_k\delta(x-x_k)\right)\cdot\overline{X},$$

where $A_k=\begin{pmatrix}0&\frac{1}{\lambda_k}\\-m_k\omega_k&0\end{pmatrix}$, $C_k=\begin{pmatrix}0&0\\-M\omega^2&0\end{pmatrix}$ with boundary conditions $P\overline{X}(x_0)+Q\overline{X}(x_n)=\overline{0}$. Cauchy matrix has the following structure

$$\widetilde{B}(x,x_0,\omega)\stackrel{def}{=}\sum_{i=0}^{n-1}\widetilde{B}_i(x,x_i,\omega)\cdot\widetilde{B}(x_i,x_0,\omega)\cdot\theta_i,$$

where $\widetilde{B}(x_i,x_0,\omega)\stackrel{def}{=}\prod_{j=0}^{i}\widetilde{C}_j\cdot\widetilde{B}_{i-j}(x_{i-j+1},x_{i-j},\omega)$, $\widetilde{C}_i=(E+C_i)$, $i=\overline{1,n-1}$, $\widetilde{B}(x_i,x_i,\omega)\stackrel{def}{=}E$,

$\widetilde{B}_i(x,s,\omega)=\begin{pmatrix}\cos\alpha_i(x-s)&\frac{\sin\alpha_i(x-s)}{\lambda_i\alpha_i}\\-\lambda_i\alpha_i\sin\alpha_i(x-s)&\cos\alpha_i(x-s)\end{pmatrix}$, $\alpha_i=\omega\sqrt{\frac{m_i}{\lambda_i}}$.

Vector $\overline{C}$ in (14) is $\begin{pmatrix}0\\1\end{pmatrix}$.

The characteristic equation of function $v(x,t)$ takes the form $b_{12}(\omega)=0$. The function $v(x,t)$ is the same as in (17).

7 A Case of Piecewise Constant Functions

Let in (1) $m(x)$, $f(x)$, $\lambda(x)$ – are piecewise constant functions:

$$m(x)=\sum_{i=0}^{n-1}m_i\theta_i(x),\ f(x)=\sum_{i=0}^{n-1}f_i\theta_i(x),\ \lambda(x)=\sum_{i=0}^{n-1}\lambda_i\theta_i(x).$$

Let's consider boundary conditions (6) with matrix $P=\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $Q=\begin{pmatrix}0&0\\1&1\end{pmatrix}$. Under these conditions, the Cauchy matrix components $b_i(x,s)=\frac{x-s}{\lambda_i}$.

Let's denote

$$\sigma_n = q_{12} \sum_{m=0}^{n-1} b_m(x_{m+1}, x_m) + q_{22},$$

$$I_{k-1}(x_k) = -\int_{x_{k-1}}^{x_k} b_{k-1}(x_k, s) f_{k-1}\, ds,$$

$$I_{k-1}^{[1]}(x_k) = -\int_{x_{k-1}}^{x_k} f_{k-1}\, ds.$$

The function $v(x,t)$ is the following

$$v(x,t) = \frac{1}{p_{11}\sigma_n - q_{21}p_{12}} \cdot (\psi_0(t)\sigma_n - p_{12}\psi_n(t) - \psi_0(t)q_{21} \cdot (b_i(x, x_i)$$

$$+ \sum_{m=0}^{i-1} b_m(x_{m+1}, x_m))) + \frac{p_{12}}{p_{11}\sigma_n - q_{21}p_{12}} \cdot (\psi_n(t) - q_{21}(\sum_{k=0}^{n} I_{k-1}(x_k)$$

$$+ I_{k-1}^{[1]}(x_k) \cdot \sum_{m=k}^{n-1} b_m(x_{m+1}, x_m)) - q_{22} \sum_{k=0}^{n} I_{k-1}^{[k]}(x_k))\cdot$$

$$(1 + b_i(x, x_i) + \sum_{m=0}^{i-1} b_m(x_{m+1}, x_m))$$

In this case the characteristic equation of the problem to eigenvalues is

$$b_{12}(\omega) + b_{22}(\omega) - b_{11}(\omega) - b_{21}(\omega) = 0.$$

Vector $\overline{C}$ in (14) is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The function $v(x,t)$ is the same as in (17).

By specifying the number of partition segments, material parameters, and core dimensions, we can obtain an analytical expression of the required number of eigenvalues and eigenfunctions. We do all the calculations in Maple package.

8 An Example of a Numerical Implementation of the Method for a Rod of Two Pieces

Modern software allows you to get the required number of eigenvalues and eigenfunctions, which ensures the appropriate accuracy of the solution. Consider the result of using the Maple package to obtain a solution to the problem. For example, consider a steel rod 1 m long, consisting of two cylindrical pieces of equal length cross-sectional area, respectively, are $F_0 = 0,0025\,\pi\text{m}^2$, $F_1 = 0,000625\,\pi\text{m}^2$, $x_0 = 0$, $x_1 = 0,5$, $x_2 = 1$. The Young's modulus for steel is

$E = 20394324259$ kg/m^2, density is $\rho = 7900$ kg/m^3. Consider the equation of longitudinal oscillations of the rod

$$\frac{\rho}{E} \cdot F(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(F(x)\frac{\partial u}{\partial x}\right)$$

with boundary conditions

$$\begin{cases} u(x_0, t) = 1, \\ u(x_2, t) = 1 \end{cases}$$

and initial conditions

$$\begin{cases} u(x, 0) = \varphi_0(x), \\ \frac{\partial u}{\partial t}(x, 0) = \varphi_1(x). \end{cases}$$

Calculations are performed in the Maple package (Figs. 1, 2, 3, 4 and 5).

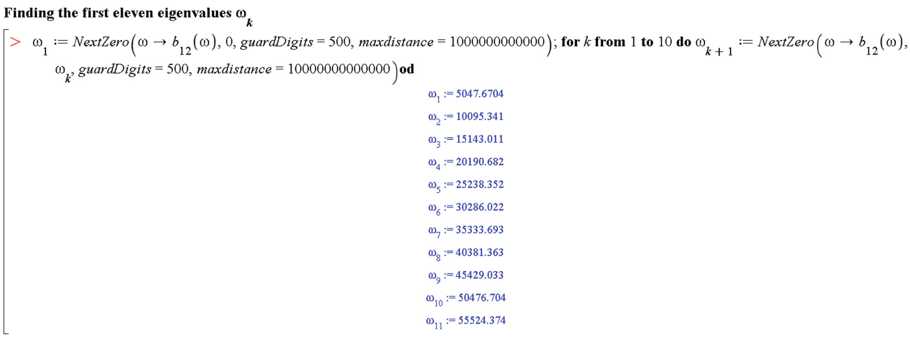

Fig. 1. Finding the first eleven eigenvalues

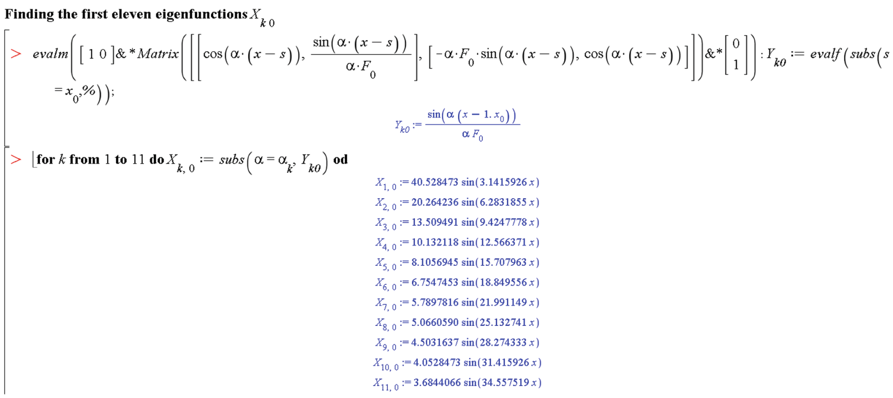

Fig. 2. Finding the first eleven eigenfunctions $X_{k,0}$

Finding the first eleven eigenfunctions X_{k1}

```
> evalm([1 0]&*Matrix([[cos(α·(x − s)), sin(α·(x − s))/(α·F_1)], [−α·F_1·sin(α·(x − s)), cos(α·(x − s))]])&*B&*[0; 1]) : Y_k1
    := evalf(subs(s = x_1, %));
Y_k1 := 1.0228719 10^6 cos(α (x − 0.5)) cos(0.00031119234 √(ω^2)) sin(0.00031119234 √(ω^2)) / √(ω^2)
    + 509.29581 sin(α (x − 0.5)) (−0.24999999 sin(0.00031119234 √(ω^2))^2 + cos(0.00031119234 √(ω^2))^2) / α
> for k from 1 to 11 do X_{k,1} := evalf(subs(α = α_k, ω = ω_k, Y_k1)) od
X_{1,1} := −0.000014834456 cos(3.1415926 x − 1.5707963) − 40.528471 sin(3.1415926 x − 1.5707963)
X_{2,1} := 0.000014834455 cos(6.2831855 x − 3.1415928) + 81.056944 sin(6.2831855 x − 3.1415928)
X_{3,1} := −0.0000013249643 cos(9.4247778 x − 4.7123889) − 13.509491 sin(9.4247778 x − 4.7123889)
X_{4,1} := 0.000014834455 cos(12.566371 x − 6.2831855) + 40.528472 sin(12.566371 x − 6.2831855)
X_{5,1} := −0.0000067287610 cos(15.707963 x − 7.8539815) − 8.1056944 sin(15.707963 x − 7.8539815)
X_{6,1} := 0.0000047023372 cos(18.849556 x − 9.4247780) + 27.018982 sin(18.849556 x − 9.4247780)
X_{7,1} := −0.000020624237 cos(21.991149 x − 10.995574) − 5.7897815 sin(21.991149 x − 10.995574)
X_{8,1} := 0.0000097683964 cos(25.132741 x − 12.566370) + 20.264237 sin(25.132741 x − 12.566370)
X_{9,1} := −0.0000013249643 cos(28.274333 x − 14.137166) − 4.5031636 sin(28.274333 x − 14.137166)
X_{10,1} := 0.000014834456 cos(31.415926 x − 15.707963) + 16.211389 sin(31.415926 x − 15.707963)
X_{11,1} := −0.0000074656420 cos(34.557519 x − 17.278760) − 3.6844066 sin(34.557519 x − 17.278760)
```

Fig. 3. Finding the first eleven eigenfunctions $X_{k,1}$

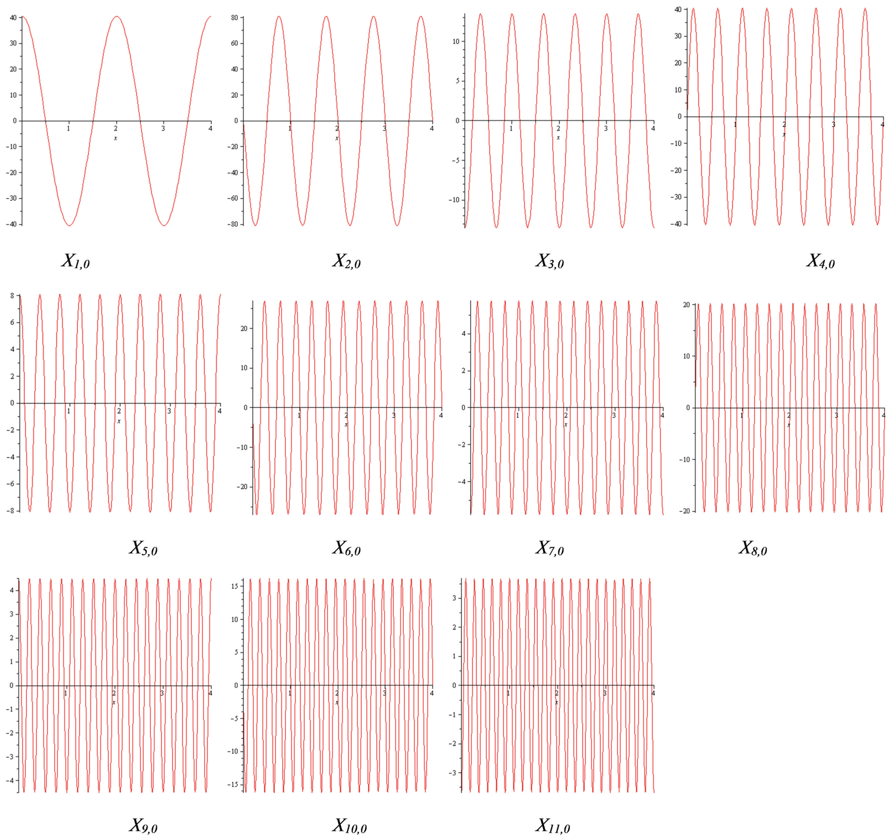

Fig. 4. Finding the first eleven eigenfunctions $X_{k,0}$

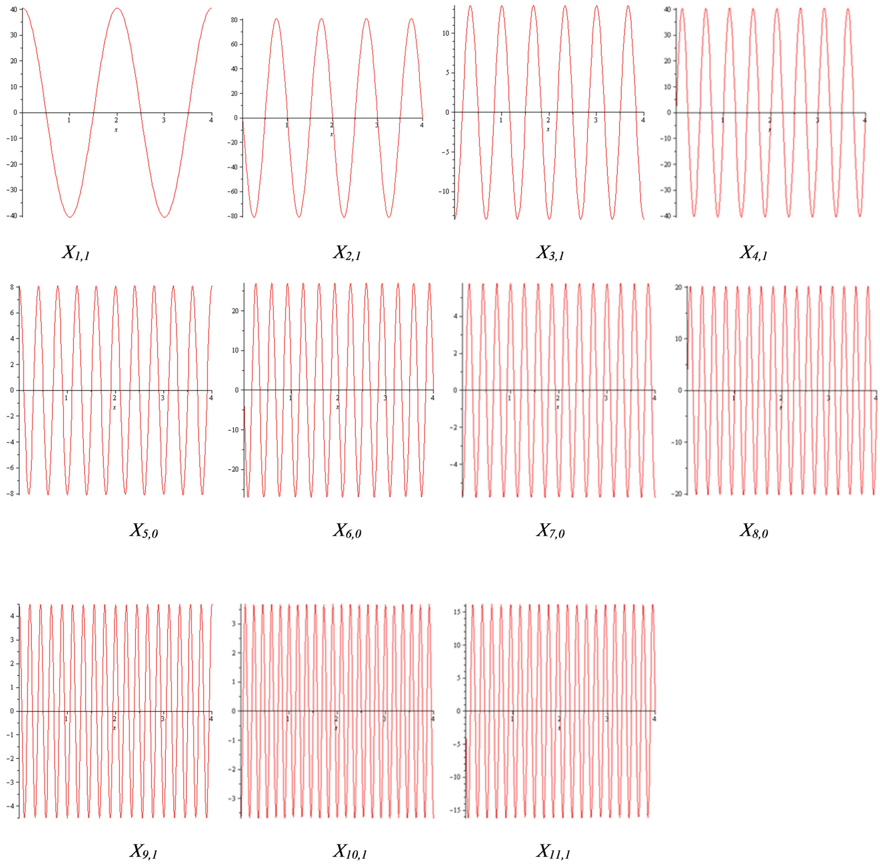

Fig. 5. Finding the first eleven eigenfunctions $X_{k,1}$

9 Conclusions

In the work on the basis of the created new method of solving nonstationary problems for equations of hyperbolic type, the solution of the actual scientific and technical problem by methods of mathematical modeling of wave processes is given. The method of solving equations in partial derivatives of the second order of the hyperbolic type described in the work makes it possible to model oscillating processes in horizontal rods consisting of an arbitrary number of pieces and having different cross sections. The proposed direct method can be used in the study of oscillatory processes without the use of approximate and operational calculus methods. The solutions of the hyperbolic equation with piecewise continuous coefficients on the spatial variable and right-hand sides with the most general local boundary conditions are obtained. The partial case, piecewise-constant coefficients and right-hand sides are singled out, when the solutions of the initial problem can be obtained in a closed form. Using the reduction method, the solutions of the general first boundary value problem for the equation of hyperbolic type with piecewise continuous coefficients and stationary inhomogeneity are obtained. The general first boundary value problem for the equation of hyperbolic type with piecewise - constant coefficients and - features is investigated. By specifying the number of partition segments, material parameters, and core dimensions, we can obtain an analytical expression of the required number of eigenvalues and eigenfunctions. We do all the calculations in Maple package. The possibilities of application of the proposed method are much wider than this work and, in particular, can be used in further research.

References

1. Al-Khaled, K., Hazaimeh, H.: Comparison methods for solving non-linear sturm-liouville eigenvalues problem. Symmetry **1179**(12), 1–17 (2020). https://doi.org/10.3390/sym12071179
2. Arsenin, V.Y.: Methods of Mathematical Physics and Special Functions, p. 432. Nauka, Moscow (1984)
3. Ashyralyev, A., Aggez, N.: Nonlocal boundary value hyperbolic problems involving integral conditions. Bound. Value Prob. (1), 1–10 (2014). https://doi.org/10.1186/s13661-014-0205-4
4. Atkinson, F.: Discrete and Continuous Boundary Value Problems, p. 518. Academic Press, Cambridge (1964)
5. Borwein, J.M., Skerritt, M.P.: An Introduction to Modern Mathematical Computing: With Maple, p. 233. Springer, New York (2011). https://doi.org/10.1007/978-1-4614-0122-3
6. Hornikx, M.: The extended fourier pseudospectral time-domain method for atmospheric sound propagation. J. Acoust. Soc. Am. **1632**(4), 1–20 (2010). https://doi.org/10.1121/1.3474234
7. Kong, O., Wu, H., Zettl, A.: Sturm-liouville problems with finite spectrum. J. Math. Anal. Appl. **263**, 748–762 (2001). https://doi.org/10.1006/jmaa.2001.7661
8. Lysenko, A., Yurkov, N., Trusov, V., Zhashkova, T., Lavendels, J.: Sum-of-squares based cluster validity index and significance analysis. Lect. Notes Comput. Sci. (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics) **5495**, 313–322 (2009). https://doi.org/10.1007/978-3-642-04921-7_32
9. Martin, N., Nilsson, P.: The moving-eigenvalue method: hitting time for ito processes and moving boundaries. J. Phys. A Math. Theoretica **53**(40), 1–32 (2020). https://doi.org/10.1088/1751-8121/ab9c59
10. Mennicken, R., Möller, M.: Non-Self-Adjoint Boundary Eigenvalue Problems, p. 518. North Holland (2003)
11. Mukhtarov, O., Yücel, M.: A study of the eigenfunctions of the singular sturm-liouville problem using the analytical method and the decomposition technique. Mathematics **415**(8), 1–14 (2020). https://doi.org/10.3390/math8030415
12. Sabitov, K.B., Zaitseva, N.V.: Initial-boundary value problem for hyperbolic equation with singular coefficient and integral condition of second kind. Lobachevskii J. Math. **39**(9), 1419–1427 (2018). https://doi.org/10.1134/S1995080218090299
13. Tatsii, R.M., Pazen, O.Y.: Direct (classical) method of calculation of the temperature field in a hollow multilayer cylinder. J. Eng. Phys. Thermophys. **91**(6), 1373–1384 (2018). https://doi.org/10.1007/s10891-018-1871-3

14. Tatsij, R.M., Chmyr, O.Y., Karabyn, O.O.: The total first boundary value problem for equation of hyperbolic type with piesewise constant coefficients and delta-singularities. Res. Math. Mech. **24**, 86–102 (2019)
15. Tichonov, A., Samarskii, A.: Equations of Mathematical Physics, chap. 2: Equations of the Hyperbolic Type, p. 777. Pergamon Press, Oxford (1990)
16. Tisseur, F., Meerbergen, K.: The quadratic eigenvalue problem. SIAM Rev. **42**(2), 235–286 (2001). https://doi.org/10.1137/S0036144500381988
17. Wyld, H.W., Powell, G.: Mathematical Methods for pPhysics, Chap. 1: Homogeneous Boundary Value Problems and Special Functions, p. 476. CRC Press, Boca Raton (2020)
18. Yang, F., Zhang, Y., Liu, X., Li, X.: The quasi-boundary value method for identifying the initial value of the space-time fractional diffusion equation. Acta Mathematica Scientia **40**(3), 641–658 (2020). https://doi.org/10.1007/s10473-020-0304-5
19. Yarka, U., Fedushko, S., Veselý, P.: The dirichlet problem for the perturbed elliptic equation. Mathematics **8**, 1–13 (2020). https://doi.org/10.3390/math8122108