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ЗМІСТ

КОРОТКІ ПОВІДОМЛЕННЯ

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RESEARCH ANNOUNCEMENTS

A. O. KURYLIAK, M. R. KURYLIAK, O. M. TRUSEVYCH

ARBITRARY RANDOM VARIABLES AND WIMAN'S INEQUALITY FOR ANALYTIC FUNCTIONS IN THE UNIT DISC

A. O. Kuryliak, M. R. Kuryliak, O. M. Trusevych. *Arbitrary random variables and Wiman's inequality for analytic functions in the unit disc*, Mat. Stud. **62** (2024), 39–45.

We consider the class $\mathcal{A}(\varphi, \beta)$ of random analytic functions in the unit disk $\mathbb{C} = \{z \colon |z| < 1\}$ of the form $f(z, \omega) = f(z, \omega_1, \omega_2) = \sum_{n=0}^{+\infty} R_n(\omega_1)\xi_n(\omega_2)a_n z^n$, where $a_n \in \mathbb{C} \colon \varlimsup_{n \to +\infty} \sqrt[n]{|a_n|} = 1$, $(R_n(\omega))$ is the Rademacher sequence, $(\xi_n(\omega))$ is a sequence of complex-valued random variables (denote by Δ_φ) such that there exists a constant $\beta > 0$ and a function $\varphi(N, \beta) \colon \mathbb{N} \times \mathbb{R}_+ \to [1; +\infty)$ non-decreasing by N and β for which

$$\left(\mathbf{E}\left(\max_{0 \leq n \leq N} |\xi_n|^\beta\right)\right)^{1/\beta} \asymp \varphi(N, \beta), \quad N \to +\infty, \quad \alpha = \varlimsup_{N \to +\infty} \frac{\ln \varphi(N, \beta)}{\ln N} < +\infty,$$

$$(\exists \gamma > 0)(\exists n_0 \in \mathbb{N}) \colon \sup\{\mathbf{E}|\xi_n|^{-\gamma} \colon n \geq n_0\} < +\infty.$$

By $\mathcal{A}_1(\varphi, \beta)$ we denote the class of random analytic functions in $\mathbb{D}$ of the form $f(z, \omega) = \sum_{n=0}^{+\infty} \xi_n(\omega)a_n z^n$, where a sequence $(\xi_n(\omega)) \in \Delta_\varphi$ and, in particular, may be not sub-gaussian and not independent. In the paper, there are proved the following statements:
Let $\delta > 0$. 1) Theorem 3: For $f \in \mathcal{A}(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ of finite logarithmic measure such that for all $r \in (r_0(\omega); 1) \backslash E$ we have with probability $p \in (0; 1)$

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r), \beta)\left((1-r)^{-2} \cdot \ln \frac{\mu_f(r)\varphi(N(r), \beta)}{(1-p)(1-r)}\right)^{1/4+\delta}.$$

2) Theorem 4: For a function $f \in \mathcal{A}_1(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ of finite logarithmic measure such that for all $r \in (r_0(\omega); 1) \backslash E$ we get with probability $p \in (0; 1)$

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r), \beta)\left((1-r)^{-2} \cdot \ln \frac{\mu_f(r)\varphi(N(r), \beta)}{(1-p)(1-r)}\right)^{1/2+\delta}.$$

1. Introduction.

Let us consider the class $\mathcal{A}$ of an analytic function f in the disc $\mathbb{D} := \{z \colon |z| < 1\}$ of the form

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n. \tag{1}$$

Let $M_f(r) = \max\{|f(z)| \colon |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n \colon n \geq 0\}$, $r > 0$, be the maximum modulus and the maximal term of series (1), respectively.

The analogues of Wiman's inequality for analytic functions in the unit disc $\mathbb{D}$ one can find in [1, 2]. From results proved in [2] follows such statement.

Theorem 1 ([2]). *Let $f \in \mathcal{A}$ be an analytic function of form* (1). *Then for eveery $\delta > 0$ there exists a set $E_f(\delta) \subset (0; 1)$ of finite logarithmic measure (f.l.m.), i.e. $\int_{E_f(\delta)} \frac{dr}{1-r} < +\infty$, such that for all $r \in (0; 1) \backslash E_f(\delta)$ we have*

$$M_f(r) \leq \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}. \tag{2}$$

For an analytic function $g(z) = \sum_{n=1}^{+\infty} \exp\{n^\varepsilon\} z^n$, $\varepsilon \in (0; 1)$, we have ([2])

$$M_g(r) \geq C \frac{\mu_g(r)}{1-r} \ln^{1/2} \frac{\mu_g(r)}{1-r} \quad (r \in [r_0, 1)), \ C > 0.$$

Therefore, inequality (2) is sharp in the class of analytic functions in the unit disc $\mathbb{D}$. But this inequality can be improved in some subclasses of random analytic functions ([3, 4, 5]).

Denote by $K(f, Y)$ the class of random analytic functions of the form

$$f(z, \omega) = \sum_{n=0}^{+\infty} a_n Y_n(\omega) z^n, \tag{3}$$

where $\{Y_n(\omega)\}$ is a sequence of random variables on the Steinhaus probability space $(\Omega, \mathcal{A}, P)$, and the sequence (a_n) satisfies the condition $\varliminf_{n \to +\infty} \sqrt[n]{|a_n|} = 1$.

Let $Y = (Y_n(\omega))$ be multiplicative system (MS) uniformly bounded by the number 1. That is, for all $n \in \mathbb{N}$ we have $|Y_n(\omega)| \leq 1$ almost surely (a.s.) and

$$\forall (j_1, j_2, \ldots, j_k) \in \mathbb{N}^k, \ 1 \leq j_1 < j_2 < \cdots < j_k\colon \ \mathbf{E}(Y_{j_1} Y_{j_2} \cdots Y_{j_k}) = 0,$$

where $\mathbf{E}\eta = \int_\Omega \eta(\omega) P(d\omega)$ is the expectation of a random variable η.

In 1997 P.V. Filevych proved the following theorem.

Theorem 2 ([3]). *Let $f(z, \omega)$ be random analytic function of the form* (3), *$Y_n \in MS$ and $|Y_n(\omega)| \leq 1$ for almost all $\omega \in [0; 1]$. Then a.s. in $K(f, Y)$ for any $\delta > 0$ there exists a set $E = E(f, \omega, \delta) \subset (0; 1)$ f.l.m. such that for all $r \in (0; 1) \backslash E$ we get*

$$M_f(r, \omega) \leq \mu_f(r) \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r)}{1-r} \right)^{1/4+\delta}. \tag{4}$$

The constant $1/4$ in the previous inequality cannot be replaced by a smaller number. This is indicated by another statement from [3].

Let $\big(R_n(\omega) \big)$ be the *Rademacher sequence*, i.e. a sequence of independent random variables defined on Steinhaus probability space $(\Omega, \mathcal{A}, P)$, such that for any $n \in \mathbb{Z}_+$ we have $\mathbb{P}\{\omega\colon R_n(\omega) = -1\} = \mathbb{P}\{\omega\colon R_n(\omega) = 1\} = \frac{1}{2}$.

Remark that for random entire function of the form $f(z, \omega) = \sum_{n=0}^{+\infty} R_n(\omega) a_n z^n$ the above-mentioned theorems from [3] are valid.

Suppose that (Z_n) is a sequence of real independent centered sub-gaussian random variables, that is for any $n \in \mathbb{Z}_+$ we have $\mathbf{E}Z_n = 0$ and there exist a constant $C_1 > 0$ such that for any $t \in [0; +\infty)$ we have $P\{\omega\colon |Z_n(\omega)| \geq t\} \leq 2\exp\big(-t^2/C_1\big)$. For such random variables we have (see [9]):

1) there exists $D > 0$ such that $\mathbf{E}(e^{\lambda_0 Z_k}) \leq e^{D\lambda_0^2}$ for any $k \in \mathbb{N}$ and all $\lambda_0 \in \mathbb{R}$;

2) for any $k \in \mathbb{N}$: $\mathbf{E}Z_k = 0$ and $\sup\{\mathbf{E}(Z_k^2)\colon k \in \mathbb{N}\} = \sup\{\mathbf{D}Z_k\colon k \in \mathbb{N}\} \leq 2D$, where $\mathbf{D}Z_k$ is the variance of random variable Z_k.

Consider the class of random functions of the form

$$K(f, \mathcal{Z}) = \left\{ f(z, \omega) = \sum_{n=0}^{+\infty} a_n Z_n(\omega) z^n \colon \omega \in [0; 1] \right\},$$

where $Z = (Z_n)$ is a sequence of real centered independent sub-gaussian random variables such that $(\exists \gamma > 0)(\exists n_0 \in \mathbb{N})\colon \sup\{\mathbf{E}|Z_n|^{-\gamma}\colon n \geq n_0\} < +\infty$. Analogues of inequality (2) for random analytic functions from class $K(f, \mathcal{Z})$ was considered in [5].

Remark, that in all above-mentioned statements about random analytic functions the obtained inequalities valid with probability 1 and only for sequences of random variables which are independent or MS and sub-gaussian in general (see also [3, 4, 5, 7, 8]).

In this regard Prof. O. B. Skaskiv formulated the following **problem:** *to obtain estimates of maximum modulus of random analytic functions: a) with probability $p \in (0; 1)$; b) in case of sequence $(Z_n(\omega))$: 1) is not sub-gaussian; 2) may not be independent.*

In this paper we give answer to all this questions. Similar question was considered for random entire functions in [6].

2. Notations. Here $\varphi(N) \asymp \psi(N)$, $N \to +\infty$, means the equivalence of functions up to constant factors. Precisely, $\varphi(N) \asymp \psi(N)$ means that there exist positive constants c, C such that the inequality $cf(N) \leq g(N) \leq Cf(N)$ holds for for all sufficiently large N.

Consider the random analytic functions of the form

$$f(z, \omega) = f(z, \omega_1, \omega_2) = \sum_{n=0}^{+\infty} R_n(\omega_1) \xi_n(\omega_2) a_n z^n, \tag{5}$$

where $a_n \in \mathbb{C}\colon \lim_{n \to +\infty} \sqrt[n]{|a_n|} = 1$, $(R_n(\omega))$ is the Rademacher sequence, $(\xi_n(\omega))$ is a sequence of complex-valued random variables (denote by Δ_φ) such that there exists a constant $\beta > 0$ and a function $\varphi(N, \beta)\colon \mathbb{N} \times \mathbb{R}_+ \to [1; +\infty)$ non-decreasing by N and β such that

$$\left(\mathbf{E}\Big(\max_{0 \leq n \leq N} |\xi_n|^\beta \Big) \right)^{1/\beta} \asymp \varphi(N, \beta), \quad N \to +\infty, \quad \alpha = \varlimsup_{N \to +\infty} \frac{\ln \varphi(N, \beta)}{\ln N} < +\infty, \tag{6}$$

$$(\exists \gamma > 0)(\exists n_0 \in \mathbb{N})\colon \sup\{\mathbf{E}|\xi_n|^{-\gamma}\colon n \geq n_0\} < +\infty. \tag{7}$$

Such class of random analytic functions we denote by $\mathcal{A}(\varphi, \beta)$. Remark that by conditions (6)–(7) radius of convergence of series (5) $R(\omega) = 1$ almost surely ([5]).

Remark, that for any sequence $(\xi_n(\omega))$ function $\psi(N, \beta) = \left(\mathbf{E}\big(\max_{0 \leq n \leq N} |\xi_n|^\beta \big) \right)^{1/\beta}$ is non-decreasing by N and β.

Also class of random analytic functions of the form $f(z, \omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) a_n z^n$ we denote by $\mathcal{A}_1(\varphi, \beta)$.

In this paper we will use the following notations.

$$W_N(r, \omega) = \sum_{n=N(r)}^{+\infty} |R_n(\omega_1)||\xi_n(\omega_2)||a_n|r^n, \quad N(r) = \left[\frac{1}{1-r} \ln \frac{\mu_f(r)}{1-r} \right]^m, \quad m = \left[\alpha + \frac{2}{\beta} \right] + 4,$$

where $[x]$ means integer part of x.

3. Main results. We obtain the asymptotic estimates for maximum modulus of functions $f \in \mathcal{A}(\varphi, \beta)$. Here sequence $(\xi_n(\omega))$ may not be sub-gaussian and may be dependent. The main result of this paper is the following theorem.

Theorem 3. *Let $\delta > 0$. For $f \in \mathcal{A}(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ of f.l.m. such that for all $r \in (r_0(\omega); 1) \backslash E$ we have with probability $p \in (0; 1)$*

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r), \beta) \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r) \varphi(N(r), \beta)}{(1-p)(1-r)} \right)^{1/4+\delta}.$$

Also we get the asymptotic estimates for maximum modulus of functions $f \in \mathcal{A}_1(\varphi, \beta)$.

Theorem 4. *Let $\delta > 0$. For $f \in \mathcal{A}_1(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ of f.l.m. such that for all $r \in (r_0(\omega); 1) \backslash E$ we get with probability $p \in (0; 1)$*

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r), \beta) \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r) \varphi(N(r), \beta)}{(1-p)(1-r)} \right)^{1/2+\delta}.$$

4. Some corollaries. If $\big(\xi_n(\omega)\big)$ is sub-exponential random variables (see [12]), i.e. there exist a constant $C_2 > 0$ such that for any $t \in [0; +\infty)$: $P(\omega \colon |Z_n(\omega)| \geq t) \leq 2 \exp\big(-\frac{t}{C_2}\big)$, and suppose that for any $n \in \mathbb{N}$ such that there exists $n \in \mathbb{N}$: $\mathbf{E}\xi_n = 0$, then we can choose $\beta = 1$ and prove that $\varphi(N, 1) = \mathbf{E}\big(\max_{0 \leq n \leq N} |\xi_n| \big) \leq C_3 \ln N$, $C_3 > 0$.

Corollary 1. *Let $\delta > 0$ and $\big(\xi_n(\omega)\big)$ is centered sub-exponential random variables. Then for $f \in \mathcal{A}(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ f.l.m. such that for all $r \in (r_0(\omega); 1) \backslash E$ we have with probability $p \in (0; 1)$*

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{1-p} \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r)}{(1-p)(1-r)} \right)^{1/4+\delta}.$$

If $\big(\xi_n(\omega)\big)$ satisfies the condition $\sup_{n \in \mathbb{N}} \mathbf{E}|\xi_n|^a < +\infty$ for some $a > 0$, then we can choose

$$\varphi(N, a) \leq C_4 N^{1/a}, \quad C_4 > 0.$$

Corollary 2. *Let $\delta > 0$ and $\big(\xi_n(\omega)\big)$ is such that $(\exists a > 0)$: $\sup_{n \in \mathbb{N}} \mathbf{E}|\xi_n|^a < +\infty$. Then for random entire function f of form (5) there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ f.l.m. such that for all $r \in (r_0(\omega); 1) \backslash E$ we obtain with probability $p \in (0; 1)$*

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{(1-p)^{1/a}(1-r)^{m/a+1/2+\delta}} \ln^{m/a+1/4+\delta} \frac{\mu_f(r)}{(1-p)(1-r)}.$$

5. Auxiliary lemmas. Similarly to [11] one can prove the following lemma.

Lemma 1. *Let $l(r)$ be a continuous increasing to $+\infty$ function on $(0; 1)$, $E \subset (0; 1)$ be a open set such that its complement $\overline{E}$ is such that $\overline{E} \cap (y; 1) \neq \varnothing$ for any $y \in (0; 1)$. Then there is an infinite sequence $0 < r_1 \leq \dots \leq r_n \uparrow 1$ $(n \to +\infty)$ such that 1) $(\forall n \in \mathbb{N})$: $r_n \notin E$; 2) $(\forall n \in \mathbb{N})$: $\ln l(r_n) \geq \frac{n}{2}$; 3) if $(r_n; r_{n+1}) \cap E \neq (r_n, r_{n+1})$, then $l(r_{n+1}) \leq el(r_n)$; 4) the set of indices, for which 3) holds, is unbounded.*

Lemma 2. *Let $f \in \mathcal{A}(\varphi, \beta)$. For any $\delta > 0$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ f.l.m. such that for all $r \in (r_0(\omega); 1) \backslash E$ we have $W_N(r, \omega) \leq \mu_f(r)$.*

Proof. Let $f_k(z) = \sum_{n=0}^{+\infty} n^k a_n z^n$, $\mathfrak{M}_{f_k}(r) = \sum_{n=0}^{+\infty} n^k |a_n| r^n$, $k \in \mathbb{N}$. For $\delta > 0$ we denote $E_1 = \big\{ r \colon r\frac{\partial}{\partial r} \ln \mathfrak{M}_f(r) > \frac{1}{1-r} \ln^{1+\delta} \mathfrak{M}_f(r), \ln \mathfrak{M}_f(r) > e \big\}$. Then

$$\int_{E_1} \frac{dr}{1-r} < \int_{E_1} \frac{dr}{r(1-r)} < \int_{E_1} \frac{\frac{\partial}{\partial r} \ln \mathfrak{M}_f(r) dr}{\ln^{1+\delta} \mathfrak{M}_f(r)} < \int_1^{+\infty} \frac{du}{u^{1+\delta}} < +\infty.$$

So, for $r \notin E_1$ we get $\mathfrak{M}_{f_1}(r) = \sum_{n=0}^{+\infty} n|a_n|r^n \leq \frac{1}{1-r}\mathfrak{M}_f(r)\ln^{1+\delta}\mathfrak{M}_f(r)$.

Also for $r \notin E_2$ $\left(\int_{E_2}(1-r)^{-1}dr < +\infty\right)$ we obtain

$$\mathfrak{M}_{f_2}(r) \leq \frac{1}{1-r}\mathfrak{M}_{f_1}(r)\ln^{1+\delta}\mathfrak{M}_{f_1}(r) \leq \frac{1}{(1-r)^{2+\delta}}\mathfrak{M}_f(r)\ln^{2+3\delta}\mathfrak{M}_f(r).$$

Similarly for $r \notin E_k$ we have

$$\mathfrak{M}_{f_k}(r) \leq \frac{1}{(1-r)^{k+\delta_2}}\mathfrak{M}_f(r)\ln^{k+\delta_2}\mathfrak{M}_f(r), \tag{8}$$

where the set E_k is a set of f.l.m., $\delta_2 > 0$.

For $n \geq N(r)$ denote $B_n = \{\omega\colon |\xi_n(\omega)|^\beta \geq n^{\alpha\beta+2+\delta_1}\}$, $\delta_1 > 0$. Then probabilities of these events we can estimate using Markov's inequality and (6). For some $C_1 > 0$ we have

$$\mathbb{P}(B_n) = \mathbb{P}\{\omega\colon |\xi_n|^\beta \geq n^{\alpha\beta+2+\delta_1}\} \leq \frac{\mathbf{E}|\xi_n|^\beta}{n^{\alpha\beta+2+\delta_1}} \leq \frac{1}{n^{\alpha\beta+2+\delta_1}}\mathbf{E}\left(\max_{0\leq k\leq n}|\xi_k|^\beta\right) \leq C_1\frac{\varphi^\beta(n,\beta)}{n^{\alpha\beta+2+\delta_1}}$$

as $r \uparrow 1$. So,

$$\sum_{n=N(r)}^{+\infty}\mathbb{P}(B_n) \leq C_1\sum_{n=N(r)}^{+\infty}\frac{\varphi^\beta(n,\beta)}{n^{\alpha\beta+2+\delta_1}} \leq C_1\sum_{n=N(r)}^{+\infty}\frac{1}{n^{2+\delta_1/2}} \leq \frac{1}{N^{1+\delta_1/3}(r)}, \quad r \uparrow 1.$$

Let $B = \bigcup_{n=N(r)}^{+\infty}B_n$. Then $\mathbb{P}(B) \leq \frac{1}{N^{1+\delta_1/3}(r)}$, $r \uparrow 1$. For $\omega \notin B$ we get

$$W_N(r,\omega) = \sum_{n=N(r)}^{+\infty}|R_n(\omega_1)||\xi_n(\omega_2)||a_n|r^n \leq \sum_{n=N(r)}^{+\infty}n^{\alpha+(2+\delta_1)/\beta}\frac{n}{N(r)}|a_n|r^n \leq$$

$$\leq \frac{1}{N(r)}\sum_{n=0}^{+\infty}n^{\alpha+1+(2+\delta_1)/\beta}|a_n|r^n \leq \frac{1}{N(r)}\sum_{n=0}^{+\infty}n^{[\alpha+2/\beta]+2}|a_n|r^n, \quad r \uparrow 1, \quad (r \notin E).$$

Then using (8), definition of $N(r)$ and Theorem 1, from [5] (in the case of $h(r) = (1-r)^{-1}$) we obtain

$$W_N(r,\omega) \leq \frac{1}{N(r)}\frac{1}{(1-r)^{[\alpha+2/\beta]+2+\delta_2}}\mathfrak{M}_f(r)\ln^{[\alpha+2/\beta]+2+\delta_2}\mathfrak{M}_f(r) \leq$$

$$\leq \frac{1}{N(r)}\frac{\mu_f(r)}{(1-r)^{[\alpha+2/\beta]+3+3\delta_2}}\ln^{[\alpha+2/\beta]+5/2+3\delta_2}\frac{\mu_f(r)}{1-r} \leq \frac{\mu_f(r)}{e}$$

as $r \uparrow 1$, $(r \notin E)$. Therefore, for $r \uparrow 1$ we obtain

$$\mathbb{P}\left\{\omega\colon \sum_{n=N(r)}^{+\infty}|R_n(\omega)||\xi_n(\omega_2)||a_n|r^n \geq \mu_f(r)/e\right\} \leq N^{-1-\delta_1/3}(r).$$

Let us choose $l(r) = \frac{\mu_f(r)}{1-r}$, and a set E and a sequence $\{r_k\}$ from Lemma 1. We put $F_k := \{\omega\colon W_N(r_k,\omega) \geq \mu_f(r_k)/e\}$. By the definition of $N(r)$ we get $P(F_k) \leq N^{-1-\delta_1/3}(r_k) \leq \ln^{-1-\delta_1/3}\frac{\mu_f(r_k)}{1-r_k} \leq k^{-1-\delta_1/3}$, thus $\sum_{k=1}^{+\infty}P(F_k) \leq \sum_{k=1}^{+\infty}k^{-1-\delta_1/3} < +\infty$. Then by Borel-Cantelli's lemma for almost all $\omega \in [0,1]$ and for $k \geq k_0(\omega)$ we obtain $W_N(r_k,\omega) < \frac{\mu_f(r_k)}{e}$.

Let $r \geq r_{k_0(\omega)}$ be an arbitrary number outside set the E, $r \in (r_p, r_{p+1})$. By Lemma 1 $\frac{\mu_f(r_{p+1})}{1-r_{p+1}} \leq e\frac{\mu_f(r_p)}{1-r_p} \leq e\frac{\mu_f(r)}{1-r}$ and then $\mu_f(r_{p+1}) \leq e\mu_f(r)$. Therefore for almost all $\omega \in [0;1]$ and $r \geq r_0(\omega)$ outside a set of f.l.m. E we have $W_N(r,\omega) < W_N(r_{p+1},\omega) \leq \mu_f(r_{p+1})/e \leq \mu_f(r)$. $\square$

6. Proofs.

Proof of Theorem 3. By Theorem 2, ω_1-almost surely there exists a set $E := E(\varepsilon, \omega, f) \subset (0; 1)$ of f.l.m. such that for all $r \in (0; 1) \setminus E$ we have

$$M_f(r, \omega) = M_f(r, \omega_1, \omega_2) \leq \mu_f(r, \omega_2)\left((1 - r)^{-2} \cdot \ln \frac{\mu_f(r, \omega_2)}{1 - r}\right)^{1/4 + \delta}. \tag{9}$$

Then by Lemma 2 we get

$$\mu_f(r, \omega_2) \leq \max\left\{\max_{0 \leq n \leq N(r)} |\xi_n(\omega_2)||a_n|r^n; \sup_{N(r) < n < +\infty} |\xi_n(\omega_2)||a_n|r^n\right\} \leq$$

$$\leq \max\left\{\max_{0 \leq n \leq N(r)} |\xi_n(\omega_2)| \cdot \mu_f(r); \mu_f(r)\right\} = \max\left\{\eta(\omega_2)\mu_f(r); \mu_f(r)\right\}, \quad r \uparrow 1, \ (r \notin E),$$

where $\eta(\omega_2) = \max_{0 \leq n \leq N(r)} |\xi_n(\omega_2)|$ is non-negative random variable. Then by Markov's inequality we obtain $P\{\omega\colon \eta^\beta(\omega) < \frac{\mathbf{E}\eta^\beta}{1-p}\} \geq p$, $P\{\omega\colon \eta(\omega) < \left(\frac{\mathbf{E}\eta^\beta}{1-p}\right)^{1/\beta}\} \geq p$.

Remark that there exist $\delta > 0$ and a set $E \subset (0; 1)$ of f.l.m. such that for all $r \in (0; 1) \setminus E$ we have $(\mathbf{E}\eta^\beta)^{1/\beta} \leq \varphi(N(r), \beta)$ and

$$\mu_f(r, \omega_2) \leq \max\left\{\left(\frac{\mathbf{E}\eta^\beta}{1 - p}\right)^{1/\beta}\mu_f(r); \mu_f(r)\right\} \leq$$

$$\leq \max\left\{\frac{\mu_f(r)}{(1 - p)^{1/\beta}}\varphi(N(r), \beta); \mu_f(r)\right\} = \frac{\mu_f(r)}{(1 - p)^{1/\beta}}\varphi(N(r), \beta). \tag{10}$$

Finally, it remains use inequalities (9) and (10). □

Proof of Corollary 1. It is enough to prove that we can choose $\beta = 1$ and $\varphi(N, 1) = \mathbf{E}\left(\max_{0 \leq n \leq N} |\xi_n|\right) \leq C_3 \ln N$, $C_3 > 0$. Remark that for sub-exponential random variables ([12, p.32]) there exists $b > 0$ such that for all $\lambda \in [0, 1/b]$ we have $\mathbf{E}(e^{\lambda|\xi_n|}) \leq e^{b\lambda}$. Then, for $\lambda = 1/b$ by Jensen's inequality we obtain

$$e^{\lambda\mathbf{E}\left(\max_{0 \leq n \leq N} |\xi_n|\right)} \leq \mathbf{E}\left(e^{\lambda \max_{0 \leq n \leq N} |\xi_n|}\right) = \mathbf{E}\left(\max_{0 \leq n \leq N} e^{\lambda|\xi_n|}\right) \leq \mathbf{E}\left(\sum_{n=0}^{N} e^{\lambda|\xi_n|}\right) \leq (N + 1)e^{b\lambda},$$

$$\lambda\mathbf{E}\left(\max_{0 \leq n \leq N} |\xi_n|\right) \leq \ln(N + 1) + b\lambda, \quad \mathbf{E}\left(\max_{0 \leq n \leq N} |\xi_n|\right) \leq \frac{\ln(N + 1)}{\lambda} + b \leq (b + 2)\ln N, \quad N \to +\infty.$$

□

Proof of Corollary 2. Here we can choose $\beta = a$. Then

$$\varphi(N, a) = \left(\mathbf{E}\left(\max\{|\xi_n|^a\colon 0 \leq n \leq N\}\right)\right)^{1/a} \leq \left(\mathbf{E}\left(\sum_{n=0}^{N} |\xi_n|^a\right)\right)^{1/a} =$$

$$= \left(\sum_{n=0}^{N} \mathbf{E}|\xi_n|^a\right)^{1/a} = (N + 1)^{1/a}(\mathbf{E}|\xi_n|^a)^{1/a} \leq C(a)N^{1/a} \ (N \to +\infty), \ C(a) > 0.$$

□

Proof of Theorem 4. By Theorem 1, there exists a set $E := E(\varepsilon, f) \subset (0; 1)$ f.l.m. such that for all $\omega \in [0; 1]$ and all $r \in (0; 1) \setminus E$ we have $M_f(r, \omega) \leq \frac{\mu_f(r, \omega)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r, \omega)}{1 - r}$.

It remans to use (10). □

REFERENCES

1. T. Kővari, *On the maximum modulus and maximal term of functions analytic in the unit disc*, J. London Math. Soc. (2), **41** (1966), 129–137. https://doi.org/10.1112/jlms/s1-41.1.129
2. N.M. Suleimanov, *Estimates of Wiman–Valiron type for power series with finite radius of convergence, and their sharpness*, DAN USSR, **253** (1980), №4, 822–824.
3. P.V. Filevych *Wiman-Valiron type estimates for random analytical functions in the unit disc*, Integral transformations and its applications to value boundary problems. Kyiv, Inst. Math., **15** (1997), 227–238.
4. O.B. Skaskiv, A.O. Kuryliak, *Direct analogues of Wiman's inequality for analytic functions in the unit disc*, Carpathian Math. Publ., **2** (2010), №1, 109–118.
5. A.O. Kuryliak, O.B. Skaskiv, *Wiman's type inequality for analytic and entire functions and h-measure of an exceptional sets*, Carpathian Math. Publ., **12** (2020), №2, 492–498.
   https://doi.org/10.15330/cmp.12.2.492-498
6. A.O. Kuryliak, O.B. Skaskiv, *Arbitrary random variables and Wiman's inequality for entire functions*, Axioms, 2024 (submited).
7. A.O. Kuryliak, O.B. Skaskiv, O.V. Zrum, *Levy's phenomenon for entire functions of several variables*, Ufa Math. J., **6** (2014), 118–127. https://doi.org/10.13108/2014-6-2-111
8. A.O. Kuryliak, L.O. Shapovalovska, O.B. Skaskiv, *Wiman's type inequality for analytic functions in the polydisc*, Ukr. Mat. J., **68** (2016), №1, 78–86.
9. J.P. Kahane, Some random series of functions, Cambridge University Press, 1985, 305 p.
   https://doi.org/10.1017/S0013091500017521
10. A. Kuryliak, *Subnormal independent random variables and Levy's phenomenon for entire functions*, Mat. Stud., **47** (2017), №1, 10–19. https://doi.org/10.15330/ms.47.1.10-19
11. P.V. Filevych, *Correlation between the maximum modulus and maximal term of random entire functions*, Mat. Stud., **7** (1997), №2, 157–166.
12. R. Vershynin, High-dimensional probability, V.47, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, 2024. https://doi.org/10.1017/9781108231596

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