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Львівський національний університет ім. Івана Франка,
механіко-математичний факультет,
Математичні студії
e-mail: math.studii@lnu.edu.ua,
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Address:

Matematychni Studii
Dept. of Mechanics and Mathematics,
Ivan Franko National University of Lviv,
1 Universytetska St.,
79000, Lviv, Ukraine,
e-mail: math.studii@lnu.edu.ua
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A. O. KURYLIAK, M. R. KURYLIAK, O. M. TRUSEVYCH

ARBITRARY RANDOM VARIABLES AND WIMAN'S INEQUALITY FOR ANALYTIC FUNCTIONS IN THE UNIT DISC

A. O. Kuryliak, M. R. Kuryliak, O. M. Trusevych. *Arbitrary random variables and Wiman's inequality for analytic functions in the unit disc*, Mat. Stud. **62** (2024), 39–45.

We consider the class $\mathcal{A}(\varphi, \beta)$ of random analytic functions in the unit disk $\mathbb{C} = \{z: |z| < 1\}$ of the form $f(z, \omega) = f(z, \omega_1, \omega_2) = \sum_{n=0}^{+\infty} R_n(\omega_1)\xi_n(\omega_2)a_n z^n$, where $a_n \in \mathbb{C}: \lim_{n \rightarrow +\infty} \sqrt[n]{|a_n|} = 1$, $(R_n(\omega))$ is the Rademacher sequence, $(\xi_n(\omega))$ is a sequence of complex-valued random variables (denote by Δ_φ) such that there exists a constant $\beta > 0$ and a function $\varphi(N, \beta): \mathbb{N} \times \mathbb{R}_+ \rightarrow [1; +\infty)$ non-decreasing by N and β for which

$$\left(\mathbf{E}\left(\max_{0 \leq n \leq N} |\xi_n|^\beta\right)\right)^{1/\beta} \asymp \varphi(N, \beta), \quad N \rightarrow +\infty, \quad \alpha = \overline{\lim}_{N \rightarrow +\infty} \frac{\ln \varphi(N, \beta)}{\ln N} < +\infty,$$

$$(\exists \gamma > 0)(\exists n_0 \in \mathbb{N}): \sup\{\mathbf{E}|\xi_n|^{-\gamma}: n \geq n_0\} < +\infty.$$

By $\mathcal{A}_1(\varphi, \beta)$ we denote the class of random analytic functions in \mathbb{D} of the form $f(z, \omega) = \sum_{n=0}^{+\infty} \xi_n(\omega)a_n z^n$, where a sequence $(\xi_n(\omega)) \in \Delta_\varphi$ and, in particular, may be not sub-gaussian and not independent. In the paper, there are proved the following statements:

Let $\delta > 0$. 1) Theorem 3: For $f \in \mathcal{A}(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ of finite logarithmic measure such that for all $r \in (r_0(\omega); 1) \setminus E$ we have with probability $p \in (0; 1)$

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r), \beta) \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r) \varphi(N(r), \beta)}{(1-p)(1-r)} \right)^{1/4+\delta}.$$

2) Theorem 4: For a function $f \in \mathcal{A}_1(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ of finite logarithmic measure such that for all $r \in (r_0(\omega); 1) \setminus E$ we get with probability $p \in (0; 1)$

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r), \beta) \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r) \varphi(N(r), \beta)}{(1-p)(1-r)} \right)^{1/2+\delta}.$$

1. Introduction. Let us consider the class \mathcal{A} of an analytic function f in the disc $\mathbb{D} := \{z: |z| < 1\}$ of the form

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n. \tag{1}$$

Let $M_f(r) = \max\{|f(z)|: |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n: n \geq 0\}$, $r > 0$, be the maximum modulus and the maximal term of series (1), respectively.

The analogues of Wiman's inequality for analytic functions in the unit disc \mathbb{D} one can find in [1, 2]. From results proved in [2] follows such statement.

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Theorem 1 ([2]). *Let $f \in \mathcal{A}$ be an analytic function of form (1). Then for every $\delta > 0$ there exists a set $E_f(\delta) \subset (0; 1)$ of finite logarithmic measure (f.l.m.), i.e. $\int_{E_f(\delta)} \frac{dr}{1-r} < +\infty$, such that for all $r \in (0; 1) \setminus E_f(\delta)$ we have*

$$M_f(r) \leq \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}. \quad (2)$$

For an analytic function $g(z) = \sum_{n=1}^{+\infty} \exp\{n^\varepsilon\} z^n$, $\varepsilon \in (0; 1)$, we have ([2])

$$M_g(r) \geq C \frac{\mu_g(r)}{1-r} \ln^{1/2} \frac{\mu_g(r)}{1-r} \quad (r \in [r_0, 1)), \quad C > 0.$$

Therefore, inequality (2) is sharp in the class of analytic functions in the unit disc \mathbb{D} . But this inequality can be improved in some subclasses of random analytic functions ([3, 4, 5]).

Denote by $K(f, Y)$ the class of random analytic functions of the form

$$f(z, \omega) = \sum_{n=0}^{+\infty} a_n Y_n(\omega) z^n, \quad (3)$$

where $\{Y_n(\omega)\}$ is a sequence of random variables on the Steinhaus probability space (Ω, \mathcal{A}, P) , and the sequence (a_n) satisfies the condition $\lim_{n \rightarrow +\infty} \sqrt[n]{|a_n|} = 1$.

Let $Y = (Y_n(\omega))$ be multiplicative system (MS) uniformly bounded by the number 1. That is, for all $n \in \mathbb{N}$ we have $|Y_n(\omega)| \leq 1$ almost surely (a.s.) and

$$\forall (j_1, j_2, \dots, j_k) \in \mathbb{N}^k, \quad 1 \leq j_1 < j_2 < \dots < j_k: \quad \mathbf{E}(Y_{j_1} Y_{j_2} \dots Y_{j_k}) = 0,$$

where $\mathbf{E}\eta = \int_{\Omega} \eta(\omega) P(d\omega)$ is the expectation of a random variable η .

In 1997 P.V. Filevych proved the following theorem.

Theorem 2 ([3]). *Let $f(z, \omega)$ be random analytic function of the form (3), $Y_n \in MS$ and $|Y_n(\omega)| \leq 1$ for almost all $\omega \in [0; 1]$. Then a.s. in $K(f, Y)$ for any $\delta > 0$ there exists a set $E = E(f, \omega, \delta) \subset (0; 1)$ f.l.m. such that for all $r \in (0; 1) \setminus E$ we get*

$$M_f(r, \omega) \leq \mu_f(r) \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r)}{1-r} \right)^{1/4+\delta}. \quad (4)$$

The constant 1/4 in the previous inequality cannot be replaced by a smaller number. This is indicated by another statement from [3].

Let $(R_n(\omega))$ be the *Rademacher sequence*, i.e. a sequence of independent random variables defined on Steinhaus probability space (Ω, \mathcal{A}, P) , such that for any $n \in \mathbb{Z}_+$ we have $\mathbb{P}\{\omega: R_n(\omega) = -1\} = \mathbb{P}\{\omega: R_n(\omega) = 1\} = \frac{1}{2}$.

Remark that for random entire function of the form $f(z, \omega) = \sum_{n=0}^{+\infty} R_n(\omega) a_n z^n$ the above-mentioned theorems from [3] are valid.

Suppose that (Z_n) is a sequence of real independent centered sub-gaussian random variables, that is for any $n \in \mathbb{Z}_+$ we have $\mathbf{E}Z_n = 0$ and there exist a constant $C_1 > 0$ such that for any $t \in [0; +\infty)$ we have $P\{\omega: |Z_n(\omega)| \geq t\} \leq 2 \exp(-t^2/C_1)$. For such random variables we have (see [9]):

- 1) there exists $D > 0$ such that $\mathbf{E}(e^{\lambda_0 Z_k}) \leq e^{D\lambda_0^2}$ for any $k \in \mathbb{N}$ and all $\lambda_0 \in \mathbb{R}$;
- 2) for any $k \in \mathbb{N}$: $\mathbf{E}Z_k = 0$ and $\sup\{\mathbf{E}(Z_k^2): k \in \mathbb{N}\} = \sup\{\mathbf{D}Z_k: k \in \mathbb{N}\} \leq 2D$, where $\mathbf{D}Z_k$ is the variance of random variable Z_k .

Consider the class of random functions of the form

$$K(f, \mathcal{Z}) = \left\{ f(z, \omega) = \sum_{n=0}^{+\infty} a_n Z_n(\omega) z^n : \omega \in [0; 1] \right\},$$

where $Z = (Z_n)$ is a sequence of real centered independent sub-gaussian random variables such that $(\exists \gamma > 0)(\exists n_0 \in \mathbb{N}) : \sup\{\mathbf{E}|Z_n|^{-\gamma} : n \geq n_0\} < +\infty$. Analogues of inequality (2) for random analytic functions from class $K(f, \mathcal{Z})$ was considered in [5].

Remark, that in all above-mentioned statements about random analytic functions the obtained inequalities valid with probability 1 and only for sequences of random variables which are independent or MS and sub-gaussian in general (see also [3, 4, 5, 7, 8]).

In this regard Prof. O. B. Skaskiv formulated the following **problem**: *to obtain estimates of maximum modulus of random analytic functions: a) with probability $p \in (0; 1)$; b) in case of sequence $(Z_n(\omega))$: 1) is not sub-gaussian; 2) may not be independent.*

In this paper we give answer to all this questions. Similar question was considered for random entire functions in [6].

2. Notations. Here $\varphi(N) \asymp \psi(N)$, $N \rightarrow +\infty$, means the equivalence of functions up to constant factors. Precisely, $\varphi(N) \asymp \psi(N)$ means that there exist positive constants c, C such that the inequality $c\varphi(N) \leq \psi(N) \leq C\varphi(N)$ holds for for all sufficiently large N .

Consider the random analytic functions of the form

$$f(z, \omega) = f(z, \omega_1, \omega_2) = \sum_{n=0}^{+\infty} R_n(\omega_1) \xi_n(\omega_2) a_n z^n, \quad (5)$$

where $a_n \in \mathbb{C}$: $\lim_{n \rightarrow +\infty} \sqrt[n]{|a_n|} = 1$, $(R_n(\omega))$ is the Rademacher sequence, $(\xi_n(\omega))$ is a sequence of complex-valued random variables (denote by Δ_φ) such that there exists a constant $\beta > 0$ and a function $\varphi(N, \beta) : \mathbb{N} \times \mathbb{R}_+ \rightarrow [1; +\infty)$ non-decreasing by N and β such that

$$\left(\mathbf{E} \left(\max_{0 \leq n \leq N} |\xi_n|^\beta \right) \right)^{1/\beta} \asymp \varphi(N, \beta), \quad N \rightarrow +\infty, \quad \alpha = \overline{\lim}_{N \rightarrow +\infty} \frac{\ln \varphi(N, \beta)}{\ln N} < +\infty, \quad (6)$$

$$(\exists \gamma > 0)(\exists n_0 \in \mathbb{N}) : \sup\{\mathbf{E}|\xi_n|^{-\gamma} : n \geq n_0\} < +\infty. \quad (7)$$

Such class of random analytic functions we denote by $\mathcal{A}(\varphi, \beta)$. Remark that by conditions (6)–(7) radius of convergence of series (5) $R(\omega) = 1$ almost surely ([5]).

Remark, that for any sequence $(\xi_n(\omega))$ function $\psi(N, \beta) = \left(\mathbf{E} \left(\max_{0 \leq n \leq N} |\xi_n|^\beta \right) \right)^{1/\beta}$ is non-decreasing by N and β .

Also class of random analytic functions of the form $f(z, \omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) a_n z^n$ we denote by $\mathcal{A}_1(\varphi, \beta)$.

In this paper we will use the following notations.

$$W_N(r, \omega) = \sum_{n=N(r)}^{+\infty} |R_n(\omega_1)| |\xi_n(\omega_2)| |a_n| r^n, \quad N(r) = \left[\frac{1}{1-r} \ln \frac{\mu_f(r)}{1-r} \right]^m, \quad m = \left[\alpha + \frac{2}{\beta} \right] + 4,$$

where $[x]$ means integer part of x .

3. Main results. We obtain the asymptotic estimates for maximum modulus of functions $f \in \mathcal{A}(\varphi, \beta)$. Here sequence $(\xi_n(\omega))$ may not be sub-gaussian and may be dependent. The main result of this paper is the following theorem.

Theorem 3. Let $\delta > 0$. For $f \in \mathcal{A}(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ of f.l.m. such that for all $r \in (r_0(\omega); 1) \setminus E$ we have with probability $p \in (0; 1)$

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r), \beta) \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r) \varphi(N(r), \beta)}{(1-p)(1-r)} \right)^{1/4+\delta}.$$

Also we get the asymptotic estimates for maximum modulus of functions $f \in \mathcal{A}_1(\varphi, \beta)$.

Theorem 4. Let $\delta > 0$. For $f \in \mathcal{A}_1(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ of f.l.m. such that for all $r \in (r_0(\omega); 1) \setminus E$ we get with probability $p \in (0; 1)$

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r), \beta) \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r) \varphi(N(r), \beta)}{(1-p)(1-r)} \right)^{1/2+\delta}.$$

4. Some corollaries. If $(\xi_n(\omega))$ is sub-exponential random variables (see [12]), i.e. there exist a constant $C_2 > 0$ such that for any $t \in [0; +\infty)$: $P(\omega: |Z_n(\omega)| \geq t) \leq 2 \exp(-\frac{t}{C_2})$, and suppose that for any $n \in \mathbb{N}$ such that there exists $n \in \mathbb{N}$: $\mathbf{E}\xi_n = 0$, then we can choose $\beta = 1$ and prove that $\varphi(N, 1) = \mathbf{E}(\max_{0 \leq n \leq N} |\xi_n|) \leq C_3 \ln N$, $C_3 > 0$.

Corollary 1. Let $\delta > 0$ and $(\xi_n(\omega))$ is centered sub-exponential random variables. Then for $f \in \mathcal{A}(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ f.l.m. such that for all $r \in (r_0(\omega); 1) \setminus E$ we have with probability $p \in (0; 1)$

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{1-p} \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r)}{(1-p)(1-r)} \right)^{1/4+\delta}.$$

If $(\xi_n(\omega))$ satisfies the condition $\sup_{n \in \mathbb{N}} \mathbf{E}|\xi_n|^a < +\infty$ for some $a > 0$, then we can choose

$$\varphi(N, a) \leq C_4 N^{1/a}, \quad C_4 > 0.$$

Corollary 2. Let $\delta > 0$ and $(\xi_n(\omega))$ is such that $(\exists a > 0)$: $\sup_{n \in \mathbb{N}} \mathbf{E}|\xi_n|^a < +\infty$. Then for random entire function f of form (5) there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ f.l.m. such that for all $r \in (r_0(\omega); 1) \setminus E$ we obtain with probability $p \in (0; 1)$

$$M_f(r, \omega) \leq \frac{\mu_f(r)}{(1-p)^{1/a}(1-r)^{m/a+1/2+\delta}} \ln^{m/a+1/4+\delta} \frac{\mu_f(r)}{(1-p)(1-r)}.$$

5. Auxiliary lemmas. Similarly to [11] one can prove the following lemma.

Lemma 1. Let $l(r)$ be a continuous increasing to $+\infty$ function on $(0; 1)$, $E \subset (0; 1)$ be a open set such that its complement \overline{E} is such that $\overline{E} \cap (y; 1) \neq \emptyset$ for any $y \in (0; 1)$. Then there is an infinite sequence $0 < r_1 \leq \dots \leq r_n \uparrow 1$ ($n \rightarrow +\infty$) such that 1) $(\forall n \in \mathbb{N})$: $r_n \notin E$; 2) $(\forall n \in \mathbb{N})$: $\ln l(r_n) \geq \frac{n}{2}$; 3) if $(r_n; r_{n+1}) \cap E \neq (r_n, r_{n+1})$, then $l(r_{n+1}) \leq el(r_n)$; 4) the set of indices, for which 3) holds, is unbounded.

Lemma 2. Let $f \in \mathcal{A}(\varphi, \beta)$. For any $\delta > 0$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0; 1)$ f.l.m. such that for all $r \in (r_0(\omega); 1) \setminus E$ we have $W_N(r, \omega) \leq \mu_f(r)$.

Proof. Let $f_k(z) = \sum_{n=0}^{+\infty} n^k a_n z^n$, $\mathfrak{M}_{f_k}(r) = \sum_{n=0}^{+\infty} n^k |a_n| r^n$, $k \in \mathbb{N}$. For $\delta > 0$ we denote $E_1 = \{r: r \frac{\partial}{\partial r} \ln \mathfrak{M}_f(r) > \frac{1}{1-r} \ln^{1+\delta} \mathfrak{M}_f(r), \ln \mathfrak{M}_f(r) > e\}$. Then

$$\int_{E_1} \frac{dr}{1-r} < \int_{E_1} \frac{dr}{r(1-r)} < \int_{E_1} \frac{\frac{\partial}{\partial r} \ln \mathfrak{M}_f(r) dr}{\ln^{1+\delta} \mathfrak{M}_f(r)} < \int_1^{+\infty} \frac{du}{u^{1+\delta}} < +\infty.$$

So, for $r \notin E_1$ we get $\mathfrak{M}_{f_1}(r) = \sum_{n=0}^{+\infty} n|a_n|r^n \leq \frac{1}{1-r}\mathfrak{M}_f(r) \ln^{1+\delta} \mathfrak{M}_f(r)$.

Also for $r \notin E_2$ ($\int_{E_2} (1-r)^{-1} dr < +\infty$) we obtain

$$\mathfrak{M}_{f_2}(r) \leq \frac{1}{1-r}\mathfrak{M}_{f_1}(r) \ln^{1+\delta} \mathfrak{M}_{f_1}(r) \leq \frac{1}{(1-r)^{2+\delta}}\mathfrak{M}_f(r) \ln^{2+3\delta} \mathfrak{M}_f(r).$$

Similarly for $r \notin E_k$ we have

$$\mathfrak{M}_{f_k}(r) \leq \frac{1}{(1-r)^{k+\delta_2}}\mathfrak{M}_f(r) \ln^{k+\delta_2} \mathfrak{M}_f(r), \quad (8)$$

where the set E_k is a set of f.l.m., $\delta_2 > 0$.

For $n \geq N(r)$ denote $B_n = \{\omega : |\xi_n(\omega)|^\beta \geq n^{\alpha\beta+2+\delta_1}\}$, $\delta_1 > 0$. Then probabilities of these events we can estimate using Markov's inequality and (6). For some $C_1 > 0$ we have

$$\mathbb{P}(B_n) = \mathbb{P}\{\omega : |\xi_n|^\beta \geq n^{\alpha\beta+2+\delta_1}\} \leq \frac{\mathbf{E}|\xi_n|^\beta}{n^{\alpha\beta+2+\delta_1}} \leq \frac{1}{n^{\alpha\beta+2+\delta_1}} \mathbf{E}\left(\max_{0 \leq k \leq n} |\xi_k|^\beta\right) \leq C_1 \frac{\varphi^\beta(n, \beta)}{n^{\alpha\beta+2+\delta_1}}$$

as $r \uparrow 1$. So,

$$\sum_{n=N(r)}^{+\infty} \mathbb{P}(B_n) \leq C_1 \sum_{n=N(r)}^{+\infty} \frac{\varphi^\beta(n, \beta)}{n^{\alpha\beta+2+\delta_1}} \leq C_1 \sum_{n=N(r)}^{+\infty} \frac{1}{n^{2+\delta_1/2}} \leq \frac{1}{N^{1+\delta_1/3}(r)}, \quad r \uparrow 1.$$

Let $B = \bigcup_{n=N(r)}^{+\infty} B_n$. Then $\mathbb{P}(B) \leq \frac{1}{N^{1+\delta_1/3}(r)}$, $r \uparrow 1$. For $\omega \notin B$ we get

$$\begin{aligned} W_N(r, \omega) &= \sum_{n=N(r)}^{+\infty} |R_n(\omega_1)| |\xi_n(\omega_2)| |a_n| r^n \leq \sum_{n=N(r)}^{+\infty} n^{\alpha+(2+\delta_1)/\beta} \frac{n}{N(r)} |a_n| r^n \leq \\ &\leq \frac{1}{N(r)} \sum_{n=0}^{+\infty} n^{\alpha+1+(2+\delta_1)/\beta} |a_n| r^n \leq \frac{1}{N(r)} \sum_{n=0}^{+\infty} n^{[\alpha+2/\beta]+2} |a_n| r^n, \quad r \uparrow 1, \quad (r \notin E). \end{aligned}$$

Then using (8), definition of $N(r)$ and Theorem 1, from [5] (in the case of $h(r) = (1-r)^{-1}$) we obtain

$$\begin{aligned} W_N(r, \omega) &\leq \frac{1}{N(r)} \frac{1}{(1-r)^{[\alpha+2/\beta]+2+\delta_2}} \mathfrak{M}_f(r) \ln^{[\alpha+2/\beta]+2+\delta_2} \mathfrak{M}_f(r) \leq \\ &\leq \frac{1}{N(r)} \frac{\mu_f(r)}{(1-r)^{[\alpha+2/\beta]+3+3\delta_2}} \ln^{[\alpha+2/\beta]+5/2+3\delta_2} \frac{\mu_f(r)}{1-r} \leq \frac{\mu_f(r)}{e} \end{aligned}$$

as $r \uparrow 1$, ($r \notin E$). Therefore, for $r \uparrow 1$ we obtain

$$\mathbb{P}\left\{\omega : \sum_{n=N(r)}^{+\infty} |R_n(\omega)| |\xi_n(\omega_2)| |a_n| r^n \geq \mu_f(r)/e\right\} \leq N^{-1-\delta_1/3}(r).$$

Let us choose $l(r) = \frac{\mu_f(r)}{1-r}$, and a set E and a sequence $\{r_k\}$ from Lemma 1. We put $F_k := \{\omega : W_N(r_k, \omega) \geq \mu_f(r_k)/e\}$. By the definition of $N(r)$ we get $P(F_k) \leq N^{-1-\delta_1/3}(r_k) \leq \ln^{-1-\delta_1/3} \frac{\mu_f(r_k)}{1-r_k} \leq k^{-1-\delta_1/3}$, thus $\sum_{k=1}^{+\infty} P(F_k) \leq \sum_{k=1}^{+\infty} k^{-1-\delta_1/3} < +\infty$. Then by Borel-Cantelli's lemma for almost all $\omega \in [0, 1]$ and for $k \geq k_0(\omega)$ we obtain $W_N(r_k, \omega) < \frac{\mu_f(r_k)}{e}$.

Let $r \geq r_{k_0(\omega)}$ be an arbitrary number outside set the E , $r \in (r_p, r_{p+1})$. By Lemma 1 $\frac{\mu_f(r_{p+1})}{1-r_{p+1}} \leq e \frac{\mu_f(r_p)}{1-r_p} \leq e \frac{\mu_f(r)}{1-r}$ and then $\mu_f(r_{p+1}) \leq e \mu_f(r)$. Therefore for almost all $\omega \in [0, 1]$ and $r \geq r_0(\omega)$ outside a set of f.l.m. E we have $W_N(r, \omega) < W_N(r_{p+1}, \omega) \leq \mu_f(r_{p+1})/e \leq \mu_f(r)$. \square

6. Proofs.

Proof of Theorem 3. By Theorem 2, ω_1 -almost surely there exists a set $E := E(\varepsilon, \omega, f) \subset (0; 1)$ of f.l.m. such that for all $r \in (0; 1) \setminus E$ we have

$$M_f(r, \omega) = M_f(r, \omega_1, \omega_2) \leq \mu_f(r, \omega_2) \left((1-r)^{-2} \cdot \ln \frac{\mu_f(r, \omega_2)}{1-r} \right)^{1/4+\delta}. \quad (9)$$

Then by Lemma 2 we get

$$\begin{aligned} \mu_f(r, \omega_2) &\leq \max \left\{ \max_{0 \leq n \leq N(r)} |\xi_n(\omega_2)| |a_n| r^n; \sup_{N(r) < n < +\infty} |\xi_n(\omega_2)| |a_n| r^n \right\} \leq \\ &\leq \max \left\{ \max_{0 \leq n \leq N(r)} |\xi_n(\omega_2)| \cdot \mu_f(r); \mu_f(r) \right\} = \max \left\{ \eta(\omega_2) \mu_f(r); \mu_f(r) \right\}, \quad r \uparrow 1, (r \notin E), \end{aligned}$$

where $\eta(\omega_2) = \max_{0 \leq n \leq N(r)} |\xi_n(\omega_2)|$ is non-negative random variable. Then by Markov's inequality we obtain $P\{\omega: \eta^\beta(\omega) < \frac{\mathbf{E}\eta^\beta}{1-p}\} \geq p$, $P\{\omega: \eta(\omega) < (\frac{\mathbf{E}\eta^\beta}{1-p})^{1/\beta}\} \geq p$.

Remark that there exist $\delta > 0$ and a set $E \subset (0; 1)$ of f.l.m. such that for all $r \in (0; 1) \setminus E$ we have $(\mathbf{E}\eta^\beta)^{1/\beta} \leq \varphi(N(r), \beta)$ and

$$\begin{aligned} \mu_f(r, \omega_2) &\leq \max \left\{ \left(\frac{\mathbf{E}\eta^\beta}{1-p} \right)^{1/\beta} \mu_f(r); \mu_f(r) \right\} \leq \\ &\leq \max \left\{ \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r), \beta); \mu_f(r) \right\} = \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r), \beta). \end{aligned} \quad (10)$$

Finally, it remains use inequalities (9) and (10). \square

Proof of Corollary 1. It is enough to prove that we can choose $\beta = 1$ and $\varphi(N, 1) = \mathbf{E} \left(\max_{0 \leq n \leq N} |\xi_n| \right) \leq C_3 \ln N$, $C_3 > 0$. Remark that for sub-exponential random variables ([12, p.32]) there exists $b > 0$ such that for all $\lambda \in [0, 1/b]$ we have $\mathbf{E}(e^{\lambda|\xi_n|}) \leq e^{b\lambda}$. Then, for $\lambda = 1/b$ by Jensen's inequality we obtain

$$e^{\lambda \mathbf{E} \left(\max_{0 \leq n \leq N} |\xi_n| \right)} \leq \mathbf{E} \left(e^{\lambda \max_{0 \leq n \leq N} |\xi_n|} \right) = \mathbf{E} \left(\max_{0 \leq n \leq N} e^{\lambda |\xi_n|} \right) \leq \mathbf{E} \left(\sum_{n=0}^N e^{\lambda |\xi_n|} \right) \leq (N+1)e^{b\lambda},$$

$$\lambda \mathbf{E} \left(\max_{0 \leq n \leq N} |\xi_n| \right) \leq \ln(N+1) + b\lambda, \quad \mathbf{E} \left(\max_{0 \leq n \leq N} |\xi_n| \right) \leq \frac{\ln(N+1)}{\lambda} + b \leq (b+2) \ln N, \quad N \rightarrow +\infty. \quad \square$$

Proof of Corollary 2. Here we can choose $\beta = a$. Then

$$\begin{aligned} \varphi(N, a) &= \left(\mathbf{E} \left(\max_{0 \leq n \leq N} \{ |\xi_n|^a : 0 \leq n \leq N \} \right) \right)^{1/a} \leq \left(\mathbf{E} \left(\sum_{n=0}^N |\xi_n|^a \right) \right)^{1/a} = \\ &= \left(\sum_{n=0}^N \mathbf{E} |\xi_n|^a \right)^{1/a} = (N+1)^{1/a} (\mathbf{E} |\xi_n|^a)^{1/a} \leq C(a) N^{1/a} \quad (N \rightarrow +\infty), \quad C(a) > 0. \end{aligned} \quad \square$$

Proof of Theorem 4. By Theorem 1, there exists a set $E := E(\varepsilon, f) \subset (0; 1)$ f.l.m. such that for all $\omega \in [0; 1]$ and all $r \in (0; 1) \setminus E$ we have $M_f(r, \omega) \leq \frac{\mu_f(r, \omega)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r, \omega)}{1-r}$.

It remains to use (10). \square

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Ivan Franko National University of Lviv
andriykuryliak@gmail.com
kuryliakmariya@gmail.com

Lviv State University of Life Safety
trusev14@gmail.com

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