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ABOUT BOREL TYPE RELATION FOR SOME POSITIVE FUNCTIONAL SERIES

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Let f be an entire transcendental function, (λ_n) be a non-decreasing to $+\infty$ sequence, $M_f(r) = \max\{|f(z)|\colon |z| = r\}$, and $\Gamma_f(r)/r = (\ln M_f(r))'_+$ be a right derivative, $r > 0$. For a regularly convergent in $\mathbb{C}$ series of the form $F(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$ the following statement is proved (Corollary 1): If condition

$$\sum_{n=1}^{\infty} \frac{1}{n\Gamma_f(\lambda_n)} < +\infty$$

holds, then the relation $\ln M_F(r) = (1 + o(1)) \ln \mu_F(r)$ holds as $r \to +\infty$ outside a set of finite logarithmic measure, where $\mu_F(r) = \max\{|a_n| M_f(r\lambda_n)\colon n \geq 0\}$, $M_F(r) = \max\{|F(z)|\colon |z| = r\}$.

Let $\lambda = (\lambda_n)$ be non-decreasing to $+\infty$ sequence. There was obtained the conditions in [1, 2] under which for series of the form

$$F(x) = \sum_{n=1}^{\infty} a_n f(\lambda_n x), \quad a_n \geq 0 \ (n \geq 0), \tag{1}$$

and for such integrals $\int_0^{+\infty} a(t) f(tx)\nu(dt)$ as some generalizations of these series, the Borel-type asymptotic relation

$$\ln F(x) = (1 + o(1)) \ln \mu_F(x) \tag{2}$$

holds as $x \to +\infty$ outside some set of finite Lebesgue measure. Here f is a positive function on $\mathbb{R}_+ = (0, +\infty)$ such that the function $y = \ln f(x)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function on $\mathbb{R}_+$, $\mu(x, F) = \max\{a_n f(x\lambda_n)\colon n \geq 0\}$. In [3], the Borel-type relation was researched for more general positive integrals of the form $\int_0^{+\infty} a(t) f(tx + \beta(t)\tau(x))\nu(dt)$, which are generalizations of series of the Taylor-Dirichlet type. In particular, the following theorem was proved in paper [1].

Theorem 1 ([1])**.** *Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function such that the function $\ln f(x)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function and a non-negative sequence (λ_n) be such that $0 \leq \lambda_n \uparrow +\infty$ $(0 \leq n \uparrow +\infty)$. If a function F represented on $(0, +\infty)$ by a series of form* (1) *and the condition*

$$\sum_{n=1}^{+\infty} \frac{1}{n \ln f(\lambda_n)} < +\infty \tag{3}$$

holds, then there exists a set $E \subset (0, +\infty)$ of finite Lebesgue measure such that asymptotic relation (2) *holds as $x \to +\infty$ $(x \to +\infty,\ x \notin E)$.*

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Sheremeta [5] considered regularly convergent series of the form

$$G(z)=\sum_{n=0}^{+\infty} b_n g(z\beta_n),$$

that is

$$\mathfrak{M}_G(r):=\sum_{n=0}^{+\infty} |b_n|M_g(r\beta_n)<+\infty$$

for all $r>0$. Here $g(z)$ is some entire function, (β_n) is a given non-negative sequence such that $\beta_n\uparrow+\infty$ $(n\uparrow+\infty)$, and $M_g(r)=\max\{|g(z)|\colon |z|=r\}$. If we now denote $F(x)=\mathfrak{M}_G(x)$, $f(x)=M_g(x)$, $\lambda_n=\beta_n$, then we obtain a series of form (1), where the function $\ln f(x)$ is logarithmicaly convex, i.e. the function $h(x):=\ln f(e^x)$ is a convex function on $\mathbb{R}$.

For the function $f(x)=e^x$ we obtain an entire Dirichlet series $F(x)=\sum_{n=1}^{\infty} a_n e^{x\lambda_n}$. Then in view of

$$\sum_{n=1}^{+\infty}\frac{1}{n\ln f(\lambda_n)}=\sum_{n=1}^{+\infty}\frac{1}{n\lambda_n}<+\infty,$$

from Theorem 1 it follows the statement of theorem from [4], which was obtained for entire Dirichlet series. Note that if the function $\ln f(x)$ is not convex, then we cannot apply the statement of Theorem 1. The following conjecture from [5] is closely related to this circumstance. Let us denote $\Gamma_f(x)=x(\ln f(x))'$.

Conjecture 1. *If*

$$\sum_{n=1}^{\infty}\frac{1}{n\Gamma_f(\lambda_n)}<+\infty, \tag{4}$$

then asymptotical relation (2) *holds as* $x\to+\infty$ *outside some exceptional set* E *such that* $\int_E \Gamma_f(x)\frac{dx}{x}<+\infty$ *for every functions* F *of form* (1).

For an every transcendental function f we get $\Gamma_f(r)=r(\ln M_f(r))'_+\nearrow+\infty$ $(r\to+\infty)$. Therefore, from condition $\int_E \Gamma_f(x)\frac{dx}{x}<+\infty$ follows that $\int_E\frac{dx}{x}<+\infty$, that is, a set E has finite Lebesgue measure.

Note that if we choose $\ln f(t)=(\ln t)^{1+\varrho}$, $\varrho>0$, then $\Gamma_f(r)=(1+\varrho)(\ln t)^{\varrho}$. Therefore, the condition (3) is weaker than the condition (4). However, there is a caveat here. Under the conditions of Sheremeta's conjecture the function $\ln f(x)$ should be considered logarithmically convex. And in the statement of Theorem 1, the function $\ln f(x)$ is convex.

Let us prove the following statement.

Theorem 2. *Let* $f_0\colon\mathbb{R}_+\to\mathbb{R}_+$ *be a differentiable function such that the function* $y=\ln f_0(x)\colon\mathbb{R}_+\to\mathbb{R}_+$ *is a convex function and the right derivative* $L(x,f_0):=(\ln f_0(x))'_+$ *strictly instreases to* $+\infty$ $(x\ge x_0)$; (β_n) *be a non-negative sequence such that* $0\le\beta_n\uparrow+\infty$ $(0\le n\uparrow+\infty)$. *If a function* F_0 *represented on* $(0,+\infty)$ *by a series of form*

$$F_0(x)=\sum_{n=0}^{\infty} a_n f_0(\beta_n+x),\quad a_n\ge 0\ (n\ge 0), \tag{5}$$

and the condition

$$\sum_{n=1}^{+\infty}\frac{1}{nL(\beta_n,f_0)}<+\infty \tag{6}$$

holds, then there exists a set $E \subset (0,+\infty)$ *of finite Lebesgue measure such that asymptotic relation* $\ln F_0(x) = (1+o(1))\ln \mu_{F_0}(x)$ *holds as* $x \to +\infty$ $(x \to +\infty,\ x \notin E)$, *where*

$$\mu_{F_0}(x) = \max\{a_n f_0(\beta_n + x)\colon n \geq 0\}.$$

Proof. Without loss of generality we can assume that $f_0(0) = 1$. We reason similarly as in papers [1–3, 6]. It is easy to see that

$$F_0(x) = \int_0^{+\infty} a(t) f_0(t+x)\, dn_\beta(t),$$

where $n_\beta(t) = \sum_{\beta_n \leq t} 1$, $a(t)$ is some non-negative dn_β-measurable function such that $a(\beta_n) = a_n$ $(n \geq 0)$ and $a(t) = 0$ for all $t \notin \{\beta_n\colon n \geq 0\}$.

For fixed $x > 0$ we put

$$G := G_x = \Big\{t > 0\colon (\ln f_0(u))'\Big|_{u=x+t} \leq 2g_0'(x)\Big\},$$

where $g_0(x) := \ln F_0(x)$. So,

$$\int_{\mathbb{R}_+\setminus G} a(t) f_0(t+x)\, dn_\beta(t) = \int_{\mathbb{R}_+\setminus G} a(t) f_0'(t+x)\Big((\ln f_0(u))'\Big|_{u=t+x}\Big)^{-1} dn_\beta(t) \leq$$
$$\leq \frac{1}{2g_0'(x)} \int_{\mathbb{R}_+\setminus G} a(t) f_0'(t+x)\, dn_\beta(t) \leq \frac{1}{2g_0'(x)} \int_{\mathbb{R}_+} a(t) f_0'(t+x)\, dn_\beta(t) = \frac{F_0(x)}{2}.$$

Hence,

$$F_0(x) = \int_G a(t) f_0(t+x)\, dn_\beta(t) + \int_{\mathbb{R}_+\setminus G} a(t) f_0(t+x)\, dn_\beta(t) \leq \int_G a(t) f(t+x)\, dn_\beta(t) + \frac{F_0(x)}{2},$$

i.e.

$$F_0(x) \leq 2\int_G a(t) f_0(t+x)\, dn_\beta(t) \leq 2\mu_0(x,F)\nu_0(2g_0'(x)), \tag{7}$$

where $\nu_0(t) := \sum_{L(\beta_n, f_0) \leq t} 1$.

Now we need the following statement ([6]): for a given non-decreasing function $\nu_0(t)\colon \mathbb{R}_+ \to \mathbb{R}_+$ the condition

$$(\exists t_0 > 0)\colon \quad \int_{t_0}^{+\infty} \frac{d\ln \nu_0(t)}{t} < +\infty, \tag{8}$$

is equivalent to the condition that there exists a continuous function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(t) \uparrow +\infty$ $(t_0 \leq t \to +\infty)$ and

$$\int_0^{+\infty} \frac{dt}{\psi(t)} < +\infty, \qquad \ln \nu_0(t) = o(\psi^{-1}(t)) \quad (t \to +\infty). \tag{9}$$

It is easy to see that condition (8) follows from condition (6).

For the function $\psi_0(t) = \psi(t)/2$ we denote a set

$$E := \{x > x_0\colon g_0'(x) \geq \psi_0(g_0(x))\}.$$

Since

$$\text{meas}(E \cap [x_0, +\infty)) = \int_E dx \le \int_E \frac{g_0'(x)}{\psi_0(g_0(x))} dx \le \int_0^{+\infty} \frac{dt}{\psi_0(t)} < +\infty, \qquad (10)$$

the set E has finite Lebesgue measure. Therefore, from inequality (7) and relation (9) we obtain

$$\ln F_0(x) \le \ln 2 + \ln \mu(x, F_0) + \ln \nu_0(2g_0'(x)) \le$$
$$\le \ln 2 + \ln \mu(x, F_0) + \ln \nu_0(\psi(g_0(x))) = \ln \mu(x, F_0) + o(g_0(x))$$

as $x \to +\infty$ ($x \notin E$). □

From Theorem 2 we immediately obtain the following corollary.

Corollary 1. *Let* (λ_n) *be non-decreasing to* $+\infty$ *a sequence and a function* F *represented by regularly convergent functional series of form* (1), *where* f *is an entire function. If condition* (4) *satisfies, then the relation* $\ln M_F(r) = (1 + o(1)) \ln \mu_F(r)$ *holds as* $r \to +\infty$ *outside a set of finite logarithmic measure, where* $\mu_F(r) = \max\{|a_n| M_f(r\lambda_n) \colon n \ge 0\}$.

Proof. Let us denote $f_0(x) = M_f(e^x)$, $F_0(x) = M_F(e^x)$, $\beta_n = \ln \lambda_n$, and consider the series of form (5). Then

$$\mu_F(e^x) \le M_F(e^x) \le F_0(x), \quad L(x, f_0) = (\ln f_0(x))'_+ = \left.\frac{d \ln M_f(r)}{d \ln r}\right|_{r=e^x} = \Gamma_f(e^x).$$

It remains to apply the statement of Theorem 2. □

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