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Identification of Dynamic Regime Transition Models Between Erlang and Exponential Distributions

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We consider a dynamic regime transition model between the Erlang and exponential distributions. The model describes a transition from the flexible Erlang distribution at small values of x to the exponential tail at large values of x.

The probability density functions are given by

$$f(x) = \lambda \exp(-\lambda x), \qquad g(x) = \frac{\mu^k}{(k-1)!} x^{k-1} \exp(-\mu x),$$

where $x \in [0, +\infty)$, $\lambda, \mu > 0$, and $k \in \mathbb{N}$.

The transition is realized using the sigmoid weighting function

$$\omega(x) = \frac{1}{1 + \exp(-r(x-a))},$$

where $a > 0$ is the transition threshold and $r > 0$ is the transition intensity.

This work generalizes the dynamic mixture model previously introduced in [1]. Although the earlier study employed a fixed second-order Erlang distribution as a component, the current formulation extends the framework to an arbitrary shape parameter $k \in \mathbb{N}$ for the Erlang constituent. Furthermore, a threshold parameter a is incorporated into the sigmoid weighting function, enabling a more flexible description of the transition between the Erlang and exponential regimes.

The unnormalized density takes the form

$$p^*(x) = \omega(x) f(x) + (1 - \omega(x)) g(x).$$

Unlike finite mixtures with constant weights, which are automatically normalized, the proposed model requires an explicit normalizing constant C, that admits a closed-form expression in terms of the Lerch transcendent:

$$C = \frac{\lambda}{r} \Phi\left(-\exp(ra), 1, \frac{\lambda}{r}\right) + 1 - \frac{\mu^k}{r^k} \Phi\left(-\exp(ra), k, \frac{\mu}{r}\right).$$

For efficient numerical evaluation of C at large argument values, the Sommerfeld asymptotic expansion of the sigmoid function near the inflection point $x = a$ is employed. The fourth-order approximation, which provides sufficient accuracy in the studied cases, is

$$C \approx 1 + \exp(-\lambda a) - \sum_{n=0}^{k-1} \frac{(\mu a)^n}{n!} \exp(-\mu a) +$$
$$+ \frac{\pi^2}{6r^2} \left(g'(a) - f'(a)\right) + \frac{7\pi^4}{360 r^4} \left(g'''(a) - f'''(a)\right).$$

To identify the optimal model parameters, the maximum likelihood estimation (MLE) method was applied. The objective function is defined as the log-likelihood function:

$$LL(C, \mu, k, \lambda, r, a) = \sum_{n=1}^{N} \ln \left(p(x_n, C, \mu, k, \lambda, r, a) \right),$$

where x_n represents the sample elements of size N.

Computational experiments were conducted on synthetic data ($N = 300$) generated by the ratio-of-uniforms method with true parameters $\lambda = 0.1$, $\mu = 0.5$, $k = 3$, $r = 1.0$, and $a = 5.0$. Maximization was performed using gradient-based optimization methods within the *CAS Maple* environment, employing a multistart strategy to ensure convergence to the global maximum. The obtained estimates $\lambda \approx 0.105$, $\mu \approx 0.552$, $k = 3$, $r \approx 0.767$, and $a \approx 5.971$ demonstrate stable convergence to the true values. Moreover, the achieved log-likelihood value ($LL = -938.766$) exceeded that of the model with the true parameters ($LL = -941.092$), indicating an effective adaptation of the algorithm to the specific statistical realization.

A comparative analysis using the corrected Akaike Information Criterion ($AIC_c = 1889.819$) and the Bayesian Information Criterion ($BIC = 1911.755$) confirmed the statistical superiority of the proposed approach over the classical alternatives. Specifically, the transition to dynamic weights significantly improved the fit quality compared to the constant-weight mixture and the single-component Erlang distribution, which fail to adequately represent the heterogeneous structure of the data (see Fig.1).

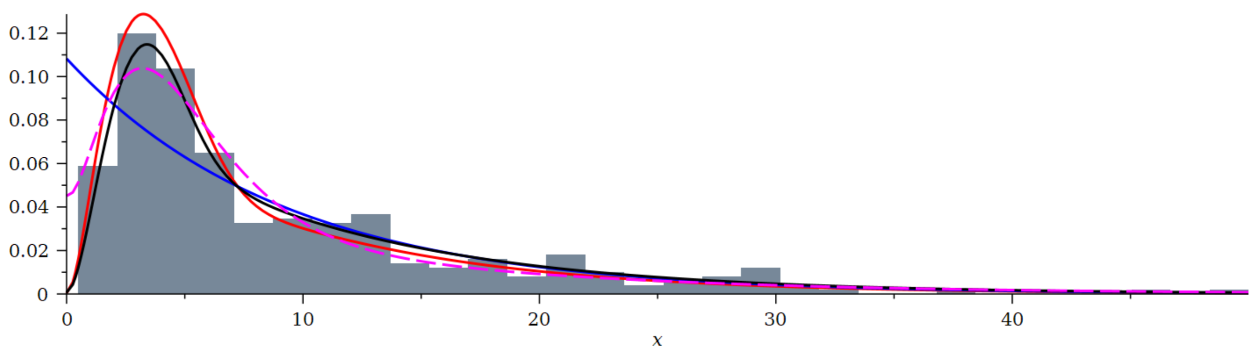

Figure 1: Approximation of the empirical distribution by various probability density functions: black line – dynamic model with true parameter values; red line – dynamic model with MLE-optimized parameters; dashed line (magenta) – mixture model with constant weights; blue line – single-component Erlang distribution.

Thus, the use of a sigmoid weighting function provides flexible model adaptation in both the mode region and the heavy-tail area, making it an efficient tool for identifying regime transitions in complex stochastic systems.

[1] Tuluchenko H. Ya., Dynamically Weighted Mixture of Exponential and Erlang Distributions with Independent Rate Parameters, *Proceedings of the XX Academician Mykhailo Kravchuk International Scientific Conference (November 17–20, 2025, Kyiv, Ukraine)*, Igor Sikorsky Kyiv Polytechnic Institute, 2025, pp. 143–144.