Ukrainian Mathematical Journal, Vol. 65, No. 10, March, 2014 (Ukrainian Original Vol. 65, No. 10, October, 2013)

SOLVABILITY OF THE FIRST BOUNDARY-VALUE PROBLEM FOR THE HEAT-CONDUCTION EQUATION WITH NONLINEAR SOURCES AND STRONG POWER SINGULARITIES

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UDC 517.95

By using the Schauder principle and the principle of contracting mappings, we study the character of point power singularities for the solution of the generalized first boundary-value problem for the heat-conduction equation with nonlinear boundary conditions. We establish sufficient conditions for the solvability of the analyzed problem.

Introduction

There are numerous works dealing with the existence and behavior of solutions of linear and semilinear parabolic equations on the boundary of a domain and at isolated points of this domain with generalized functions given on the boundary (see, e.g., [1] and the references therein and [2, 3]).

Nonlinear elliptic boundary-value problems with functions with strong power singularities given on the boundary were studied in [5]. Later, the investigations of nonlinear boundary-value problems for the heat-conduction equation in terms of generalized functions were continued on the basis of the results of these studies. Sufficient conditions for the solvability of the nonlinear generalized first boundary-value problem for the heat-condition equation in the space of functions with point singularities were established in [5]. The character of point singularities of the solution of this problem was analyzed in [6].

It is of interest to answer the question whether it is possible to consider these problems with nonlinear boundary conditions. The boundary-value problems for the heat-conduction equation with nonlinear boundary conditions were investigated in [7–11]. The well-posed solvability of the boundary-value problem for the heat-conduction equation with nonlinear boundary conditions and measures as initial data was established in [11].

In [12, 13], the Cauchy problem for a nonlinear parabolic equation was considered and the global existence and blow-up of the solutions of this problem for a finite period of time were established. The critical exponent of nonlinearity separating the domains of existence and nonexistence of the global positive solution of the Cauchy problem was determined. Similar investigations were performed by numerous researchers for (linear and nonlinear) equations of various types with linear and nonlinear boundary conditions (see, e.g., [9, 10, 14]). It was shown that, in some cases, the blow-ups may occur at isolated points or only on the boundary of the domain. It is known that the critical exponent depends on the class of functions used for the determination of the solution.

In the present paper, we study the first boundary-value problem for the heat-conduction equation in a bounded cylindrical domain in the presence of nonlinear point sources. We prove the existence of a solution of this problem in a class of functions with strong power singularities depending on the exponents of nonlinearity in the equation and boundary conditions, including, in particular, the case of Fujita "critical exponent." To prove solvability, we reduce this problem to the integral equation in a certain weight function space and apply the Schauder theorem on fixed point and the principle of contracting mappings.

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Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 65, No. 10, pp. 1388–1407, October, 2013. Original article submitted September 18, 2012; revision submitted May 13, 2013.

0041-5995/14/6510–1542

1. Basic Notation and Statement of the Problem

Let $n \in \mathbb{N}$, let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $S = \partial\Omega$ of the class C^∞, let $Q = \Omega \times (0, T]$, let $\Sigma = S \times (0, T]$, and let $0 < T < +\infty$.

We use the following notation:

$$\|x - y\| = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

is the Euclidean distance in $\mathbb{R}^n$, $P = (x, t)$, $M = (y, \tau)$, $\widehat{P} = (\hat{x}, \hat{t})$,

$$d(P, M) = |PM| = d(x, t; y, \tau) = \sqrt{\|x - y\|^2 + |t - \tau|}$$

is the parabolic distance in $\mathbb{R}^{n+1}$, η is a multiindex with components $(\eta_1, \ldots, \eta_n)$, $\eta_i \in \mathbb{Z}_+$, $i = \overline{1, n}$, $|\eta| = \eta_1 + \ldots + \eta_n$ is the length of the multiindex η, and

$$D^\eta \equiv D_x^\eta = \frac{\partial^{|\eta|}}{\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n}}.$$

Let $\varepsilon_0 > 0$ be a given number such that the surface S_{ε_0} parallel to the surface S belongs to the class C^∞. In what follows, we assume that $\varepsilon_0 \leq 1$. By $\widetilde{\varrho}(\sigma)$ we denote an infinitely differentiable nonnegative function of the order σ as $\sigma \to 0$. For any fixed point $\widehat{P} \in \overline{Q}$, we introduce a function ϱ_0 of the point $P \in \overline{Q}$ such that $0 < \varrho_0(P, \widehat{P}) \leq 1$ and

$$\varrho_0(P, \widehat{P}) = \begin{cases} \widetilde{\varrho}(|P\widehat{P}|), & |P\widehat{P}| < \dfrac{\varepsilon_0}{2}, \\ 1, & |P\widehat{P}| \geq \varepsilon_0. \end{cases}$$

Let

$$D(\overline{Q}) = C^\infty(\overline{Q}), \qquad D(\overline{\Sigma}) = C^\infty(\overline{\Sigma}), \qquad D(\overline{\Omega}) = C^\infty(\overline{\Omega}),$$

$$D^0(\overline{Q}) = \left\{ \varphi \in D(\overline{Q}) \colon \frac{\partial^k}{\partial t^k} \varphi \mid_{t=T} = 0, \ k = 0, 1, \ldots \right\},$$

$$D^0(\overline{\Sigma}) = \left\{ \varphi \in D(\overline{\Sigma}) \colon \frac{\partial^k}{\partial t^k} \varphi \mid_{t=T} = 0, \ k = 0, 1, \ldots \right\},$$

$$D_0(\overline{\Omega}) = \left\{ \varphi \in D(\overline{\Omega}) \colon \varphi|_S = 0 \right\},$$

and let ν be the unit vector of the outer normal to S.

Further, let $\left(D^0(\overline{\Sigma})\right)'$ and $\left(D_0(\overline{\Omega})\right)'$ denote the spaces of linear continuous functionals on the spaces of functions $D^0(\overline{\Sigma})$ and $D_0(\overline{\Omega})$, respectively, let $(\varphi, F)_1$ denote the value of the generalized function $F \in \left(D^0(\overline{\Sigma})\right)'$ on the test function $\varphi \in D^0(\overline{\Sigma})$, and let $(\varphi, F)_2$ denote the value of $F \in \left(D_0(\overline{\Omega})\right)'$ on $\varphi \in D_0(\overline{\Omega})$.

Assumption 1. *Let* $(\hat{x}, \hat{t}) \in \Sigma$. *Assume that*